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A New Uniform Fourth Order One-Third Step Continuous Block Method for the Direct Solutions of $y'' = f(x, y, y')$

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Authors' contributions

This work was carried out in collaboration between both authors. Author EAA proposed and derived the method, Author MAR wrote a Matlab code for implementing the method. Both authors drafted the manuscript, and managed literature searches. Both authors read and approved the final manuscript.

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Abstract

In this study, we applied the approach of collocation and interpolation to develop a new fourth order continuous one-third hybrid block method for the solutions of general second order initial value problems of ordinary differential equations. Three discrete schemes were derived from the continuous schemes. The discrete method was analyzed based on the properties of linear multistep methods and the method is found to be zero-stable, consistent and convergent. We reported an improved performance of the new method over the existing methods in the literature by solving four numerical examples and the approximate solutions obtained confirmed the superiority of our new developed scheme when compared with some latest existing approaches.

Keywords: Continuous hybrid block methods; second order initial values problems; collocation and interpolation; approximate solutions; zero stability.

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1 Introduction

The general second order initial value problems of ordinary differential equations is given as follow:

$$y'' = f(x, y, y'), x\epsilon[a, b],\ y(x_0) = y_0,\ y'(x_0) = z_0, \tag{1.1}$$

where x_0, is the initial point, y_0 is the solution at x_0, f is continuous function within the close interval ($[a, b]$).

Many researchers have solved equation (1.1) conventionally by reducing it to a system of first order ordinary differential equations before they could be able to give solution to the problem. The disadvantages of this techniques had been presented by many scholars among them are [1], [2], [3] and [4].

Several approaches were also reported in the literature for solving equation (1.1) directly without changing it to a system of first order differential equations. For example, [5], [6], [7], [8], [9], [10] and [11] applied linear multistep methods, particularly an implicit methods to solve equation (1.1) directly but with the rigor of developing separate predictors are needed which reduced the accuracy of this method.

Different authors such as [12], [13] [14], [15], [16], [17] and [18] have applied hybrid methods to solve equation (1.1) but their solutions have lower order of accuracy.

The main aim of this paper is to develop block method that gives solution to second order Initial Value Problems (IVP) directly without reducing to systems of first order ordinary differential equation and the objectives are stated below:

1. To develop a continuous implicit scheme that give solutions to second order differential equation.
2. To derive discrete scheme from the continuous implicit scheme.
3. To develop a block that solves second order differential equations.
4. To analyze the basic properties of the develop block which include consistency, zero stability, convergence, order and error constant.

In order to achieve the aim and objectives stated above, we shall interpolate, collocate and evaluate a power series approximate solution at some chosen grid and off grid points to generate an implicit continuous hybrid multistep method and we will use the new proposed method to give solutions to any problems in form of equation (1.1).

2 The Derivation of a New One-Third Step Method

In this section, we use the simple power series as an approximate solution to be of the form:

$$Y(x) = \sum_{j=0}^{i+c-1} a_j x^j, \tag{2.1}$$

where c represents collocation point and c is the interpolation point.

The second derivative of equation (2.1) gives

$$Y''(x) = \sum_{j=0}^{i+c-1} a_j j(j-1)x^{j-2} = f(x, y, y'). \tag{2.2}$$

Equations (2.1) and (2.2) are interpolated and collocated at the points $x_{n+i}, i = 0, \frac{2}{9}$ and $x_{n+c}, c = 0, \frac{1}{9}, \frac{2}{9}$, and $\frac{1}{3}$ to get a system of equation of the form

$$AX = B, \tag{2.3}$$

where $A = [a_0, a_1, a_2, a_3, a_4, a_5]^T, B = \left[y_n, y_{n+\frac{2}{9}}, f_n, f_{n+\frac{1}{9}}, f_{n+\frac{2}{9}}, f_{n+\frac{1}{3}}\right]^T$ and

$$X = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_{n+\frac{2}{9}} & x^2_{n+\frac{2}{9}} & x^3_{n+\frac{2}{9}} & x^4_{n+\frac{2}{9}} & x^5_{n+\frac{2}{9}} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{9}} & 12x^2_{n+\frac{1}{9}} & 20x^3_{n+\frac{1}{9}} \\ 0 & 0 & 2 & 6x_{n+\frac{2}{9}} & 12x^2_{n+\frac{2}{9}} & 20x^3_{n+\frac{2}{9}} \\ 0 & 0 & 2 & 6x_{n+\frac{1}{3}} & 12x^2_{n+\frac{1}{3}} & 20x^3_{n+\frac{1}{3}} \end{bmatrix}$$

By Simplifying some notation in equation (2.3) and solve for $a'_j s$, $j = 0, 1, 2, 3, 4$ and 5 and substituting the value of $a'_j s$ into the equation (2.1) gives a continuous implicit hybrid multistep method of the form :

$$Y(x) = \alpha_0(x)y_n + \alpha_{\frac{2}{9}}(x)y_{n+\frac{2}{9}} + h^2\left[\sum_{j=0}^{\frac{1}{3}} \beta_j(x)f_{n+j}\right], \tag{2.4}$$

where $j = 0, \frac{1}{9}, \frac{2}{9}$, and $\frac{1}{3}$, α_j and β_j represent continuous coefficients, $y_{n+j} = y\,(x_n + jh)$ represents numerical solution of the analytical solution at the point x_{n+j} and $f_{n+j} = f\left(x_{n+j}, y_{n+j}, y'_{n+j}\right)$.

Using the transformation

$$p = \frac{(x - x_n)}{h}. \tag{2.5}$$

The coefficients α_j and β_j are given as :

$$\begin{aligned} \alpha_0(p) &= 1 - \frac{9}{2}p \\ \alpha_{\frac{2}{9}}(p) &= \frac{9}{2}p \\ \beta_0(p) &= -\frac{1}{3240}h^2p\left(19\,683p^4 - 21\,870p^3 + 8910p^2 - 1620p + 112\right) \\ \beta_{\frac{1}{9}}(p) &= \frac{1}{1080}h^2p\left(19\,683p^4 - 18\,225p^3 + 4860p^2 - 88\right) \\ \beta_{\frac{2}{9}}(p) &= -\frac{1}{1080}h^2p\left(19\,683p^4 - 14\,580p^3 + 2430p^2 - 8\right) \\ \beta_{\frac{1}{3}}(p) &= \frac{1}{3240}h^2p\left(19\,683p^4 - 10\,935p^3 + 1620p^2 - 8\right). \end{aligned} \tag{2.6}$$

Evaluating equation (2.6) at $p = \frac{1}{9}$ and $p = \frac{1}{3}$, we get

$$y_{n+\frac{1}{9}} - \frac{1}{2}y_{n+\frac{2}{9}} - \frac{1}{2}y_n = \frac{-1}{1944}h^2\left(f_n + 10f_{n+\frac{1}{9}} + f_{n+\frac{2}{9}}\right), \tag{2.7}$$

$$y_{n+\frac{1}{3}} - \frac{3}{2}y_{n+\frac{2}{9}} + \frac{1}{2}y_n = \frac{1}{1944}h^2\left(f_n + 2f_{n+\frac{1}{3}} + 12f_{n+\frac{1}{9}} + 21f_{n+\frac{2}{9}}\right). \tag{2.8}$$

The first derivative of the equation (2.6) gives

$$\begin{aligned}
\alpha_0'(p) &= -\frac{9}{2h}\\
\alpha_{\frac{2}{9}}'(p) &= \frac{9}{2h}\\
\beta_0'(p) &= -\frac{1}{3240}h\left(-3240p + 26730p^2 - 87480p^3 + 98415p^4 + 112\right)\\
\beta_{\frac{1}{9}}'(p) &= \frac{1}{1080}h\left(14580p^2 - 72900p^3 + 98\,415p^4 - 88\right)\\
\beta_{\frac{2}{9}}'(p) &= -\frac{1}{1080}h\left(7290p^2 - 58320p^3 + 98415p^4 - 8\right)\\
\beta_{\frac{1}{3}}'(p) &= \frac{1}{3240}h\left(4860p^2 - 43740p^3 + 98415p^4 - 8\right).
\end{aligned} \tag{2.9}$$

By evaluating equation (2.9) at points $p = 0, \frac{1}{9}, \frac{2}{9}$ and $\frac{1}{3}$, we obtain

$$hy_n^{'} = -\frac{9}{2}y_n + \frac{9}{2}y_{n+\frac{2}{9}} - \frac{1}{405}h^2\left(14f_n + f_{n+\frac{1}{3}} + 33f_{n+\frac{1}{9}} - 3f_{n+\frac{2}{9}}\right), \tag{2.10}$$

$$hy_{n+\frac{1}{9}}^{'} = -\frac{9}{2}y_n + \frac{9}{2}y_{n+\frac{2}{9}} + \frac{1}{3240}h^2\left(23f_n + 7f_{n+\frac{1}{3}} + 21f_{n+\frac{1}{9}} - 51f_{n+\frac{2}{9}}\right), \tag{2.11}$$

$$hy_{n+\frac{2}{9}}^{'} = -\frac{9}{2}y_n + \frac{9}{2}y_{n+\frac{2}{9}} + \frac{1}{405}h^2\left(f_n - f_{n+\frac{1}{3}} + 27f_{n+\frac{1}{9}} + 18f_{n+\frac{2}{9}}\right), \tag{2.12}$$

$$hy_{n+\frac{1}{3}}^{'} = -\frac{9}{2}y_n + \frac{9}{2}y_{n+\frac{2}{9}} + \frac{1}{3240}h^2\left(23f_n + 127f_{n+\frac{1}{3}} + 141f_{n+\frac{1}{9}} + 429f_{n+\frac{2}{9}}\right). \tag{2.13}$$

3 Derivation of Block for a New One-Third Step Hybrid Method

In order to get block methods, the derivatives of the block method and to test for zero stability, we combine equations (2.7), (2.8) and (2.10) and we use their coefficients to form a block of the form

$$\begin{aligned}
&\begin{bmatrix} 0 & -\frac{3}{2} & 1 \\ 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{9}{2} & 0 \end{bmatrix}
\begin{bmatrix} y_{n+\frac{1}{9}} \\ y_{n+\frac{2}{9}} \\ y_{n+\frac{1}{3}} \end{bmatrix}
= \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{9}{2} \end{bmatrix}
\begin{bmatrix} y_{n-\frac{1}{9}} \\ y_{n-\frac{2}{9}} \\ y_n \end{bmatrix}
+ h \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\begin{bmatrix} y'_{n-\frac{1}{9}} \\ y'_{n-\frac{2}{9}} \\ y'_n \end{bmatrix}\\
&+h^2 \begin{bmatrix} 0 & 0 & \frac{1}{1944} \\ 0 & 0 & -\frac{1}{1944} \\ 0 & 0 & -\frac{14}{405} \end{bmatrix}
\begin{bmatrix} f_{n-\frac{1}{9}} \\ f_{n-\frac{2}{9}} \\ f_n \end{bmatrix}
+ h^2 \begin{bmatrix} \frac{12}{1944} & \frac{21}{1944} & \frac{2}{1944} \\ -\frac{10}{1944} & -\frac{1}{1944} & 0 \\ -\frac{33}{405} & \frac{3}{405} & -\frac{1}{405} \end{bmatrix}
\begin{bmatrix} f_{n+\frac{1}{9}} \\ f_{n+\frac{2}{9}} \\ f_{n+\frac{1}{3}} \end{bmatrix}
\end{aligned} \tag{3.1}$$

After normalizing the equation (3.1), we obtain

$$\begin{aligned}
&\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} y_{n+\frac{1}{9}} \\ y_{n+\frac{2}{9}} \\ y_{n+\frac{1}{3}} \end{bmatrix}
= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} y_{n-\frac{1}{9}} \\ y_{n-\frac{2}{9}} \\ y_n \end{bmatrix}
+ h \begin{bmatrix} 0 & 0 & \frac{1}{9} \\ 0 & 0 & \frac{2}{9} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}
\begin{bmatrix} y'_{n-\frac{1}{9}} \\ y'_{n-\frac{2}{9}} \\ y'_n \end{bmatrix}\\
&+h^2 \begin{bmatrix} 0 & 0 & \frac{97}{29\,160} \\ 0 & 0 & \frac{28}{3645} \\ 0 & 0 & \frac{13}{1080} \end{bmatrix}
\begin{bmatrix} f_{n-\frac{1}{9}} \\ f_{n-\frac{2}{9}} \\ f_n \end{bmatrix}
+ h^2 \begin{bmatrix} \frac{19}{4860} & -\frac{13}{9720} & \frac{1}{3645} \\ \frac{22}{1215} & -\frac{2}{1215} & \frac{2}{3645} \\ \frac{1}{30} & \frac{1}{120} & \frac{1}{540} \end{bmatrix}
\begin{bmatrix} f_{n+\frac{1}{9}} \\ f_{n+\frac{2}{9}} \\ f_{n+\frac{1}{3}} \end{bmatrix}.
\end{aligned} \tag{3.2}$$

By rewriting equation (3.2) explicitly, we get:

$$y_{n+\frac{1}{9}} = y_n + \frac{1}{9}hy_n^{'} + \frac{1}{29\,160}h^2\left(97f_n + 8f_{n+\frac{1}{3}} + 114f_{n+\frac{1}{9}} - 39f_{n+\frac{2}{9}}\right), \tag{3.3}$$

$$y_{n+\frac{2}{9}} = y_n + \frac{2}{9}hy_n^{'} + \frac{2}{3645}h^2\left(14f_n + f_{n+\frac{1}{3}} + 33f_{n+\frac{1}{9}} - 3f_{n+\frac{2}{9}}\right), \tag{3.4}$$

$$y_{n+\frac{1}{3}} = y_n + \frac{1}{9}hy_n^{'} + \frac{1}{1080}h^2\left(13f_n + 2f_{n+\frac{1}{3}} + 36f_{n+\frac{1}{9}} + 9f_{n+\frac{2}{9}}\right). \tag{3.5}$$

Substituting $y_{n+\frac{2}{9}}$ of equation (3.4) into the equations (2.11), (2.12) and (2.13) gives

$$y_{n+\frac{1}{9}}^{'} = y_n^{'} + \frac{1}{216}h\left(9f_n + f_{n+\frac{1}{3}} + 19f_{n+\frac{1}{9}} - 5f_{n+\frac{2}{9}}\right), \tag{3.6}$$

$$y_{n+\frac{2}{9}}^{'} = y_n^{'} + \frac{1}{27}h\left(f_n + 4f_{n+\frac{1}{9}} + f_{n+\frac{2}{9}}\right), \tag{3.7}$$

$$y_{n+\frac{1}{3}}^{'} = y_n^{'} + \frac{1}{24}h\left(f_n + f_{n+\frac{1}{3}} + 3f_{n+\frac{1}{9}} + 3f_{n+\frac{2}{9}}\right). \tag{3.8}$$

4 Analysis of the Properties of a New One-Third Step Method

In this section, we consider the analysis of the basic properties of a new one-third step method which includes: zero stability, region of absolute stability, order, error constants, consistency and convergence of the method.

4.1 Order and error constant of the block method

Following the method presented by [19], we define the linear difference operator as follow:

$$L[y(x);h] = \sum_{j=0}^{k}\left[\alpha_j y\left(x+jh\right) - h^2\beta_j y''\left(x+jh\right)\right]. \tag{4.1}$$

If we assume that $y(x)$ has higher derivatives, we can expand the term in equation (3.4) as a Taylor series about the point x to get the expansion:

$$L[y(x);h] = C_0y(x) + C_1hy'(x) + C_1h^2y''(x)... + C_qh^qy^q(x), \tag{4.2}$$

where

$$C_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q\alpha_j - q(q-1)\sum_{j=1}^{k} j^{q-2}\beta_j\right], \text{ where } q = 0,1,2,...n. \tag{4.3}$$

Definition: The linear difference operator and the associated block method are said to be of order p if $C_0 = C_1 = ... = C_p = C_{p+1} = 0, C_{p+2} \neq 0$. C_{p+2} is called the error constant.

By Carrying out Taylor series expansion on equations (3.3), (3.4), (3.5), (3.6), (3.7) and (3.8) to get the order of our new proposed block methods as $(4,4,4,4,4,4)$ and error constants as $\left(-\frac{7}{255091680}, -\frac{1}{15943230}, -\frac{1}{9447840}, -\frac{19}{42515280}, -\frac{19}{5314410}, -\frac{19}{1574640}\right)$.

4.2 Zero stability of the block method

In order to test for zero stability of the block method, we consider the matrix difference equation of the form:

$$P^0Y_{m+1} = P^1Y_m + h^2\left[Q^0F_{m+1} + Q^1F_m + hR^1\triangle_m\right], \tag{4.4}$$

where $Y_{m+1} = \left[y_{n+\frac{1}{9}},...,y_{n+\frac{1}{3}}\right]^T$, $Y_m = \left[y_{n-\frac{1}{9}},...,y_n\right]^T$, $F_{m+1} = \left[f_{n+\frac{1}{9}},...,f_{n+\frac{1}{3}}\right]^T$, $F_m = \left[f_{n-\frac{1}{9}},...,f_n\right]^T$, $\triangle_m = \left[\mathbb{k}_{n-\frac{1}{9}},...,\mathbb{k}_n\right]^T$, $m = 0,1,....$ The matrices P^0, P^1, Q^0, Q^1 and R^0 are the coefficients of equation (3.2) which defined as follows:

$$P^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q^0 = \begin{bmatrix} \frac{19}{4860} & -\frac{13}{9720} & \frac{1}{3645} \\ \frac{22}{1215} & -\frac{2}{1215} & \frac{2}{3645} \\ \frac{1}{30} & \frac{1}{120} & \frac{1}{540} \end{bmatrix}$$

$$Q^1 = \begin{bmatrix} 0 & 0 & \frac{97}{3240} \\ 0 & 0 & \frac{28}{405} \\ 0 & 0 & \frac{13}{120} \end{bmatrix}$$

$$R^1 = \begin{bmatrix} 0 & 0 & \frac{1}{9} \\ 0 & 0 & \frac{2}{9} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

A block method is said to be zero stable if the roots $\left|\left[\lambda P^{(0)} - P^{(1)}\right]\right| = 0$

To show that $\left|\left[\lambda P^{(0)} - P^{(1)}\right]\right| = 0$, we have:

$$\left|\left[\lambda P^{(0)} - P^{(1)}\right]\right| = \left|\left[\lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]\right| = 0$$

This implies that $\lambda^3 - \lambda^2 = 0, \lambda = 0, 0, 1$

Following Jator [3], the new developed one-step block method is zero-stable, since root (λ) has modulus less than or equal to one and $|\lambda| = 1$ is simple.

4.3 Region of absolute stability of the block method

By following Ibijola et al. [20], we formulate the stability matrix as follow:

$$M(z) = V + zB(M - zA)^{-1}U, \tag{4.5}$$

and the stability function

$$p(\eta, z) = det(\eta I - M(z)). \tag{4.6}$$

Hence, we represent the block method of the equation (3.2) in form of

$$\begin{bmatrix} Y \\ --- \\ Y_{i+1} \end{bmatrix} = \begin{bmatrix} A & & U \\ --- & --- & --- \\ B & & V \end{bmatrix} \begin{bmatrix} h^2 f(y) \\ --- \\ Y_{i-1} \end{bmatrix}. \tag{4.7}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{97}{29160} & \frac{19}{4860} & -\frac{13}{9720} & \frac{1}{3645} \\ \frac{28}{3645} & \frac{22}{1215} & -\frac{2}{1215} & \frac{2}{3645} \\ \frac{13}{1080} & \frac{1}{30} & \frac{1}{20} & \frac{1}{540} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{97}{29160} & \frac{19}{4860} & -\frac{13}{9720} & \frac{1}{3645} \\ \frac{28}{3645} & \frac{22}{1215} & -\frac{2}{1215} & \frac{2}{3645} \end{bmatrix} \quad V = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \tag{4.8}$$

$$Y = \begin{bmatrix} y_n \\ y_{n+\frac{1}{9}} \\ y_{n+\frac{2}{9}} \\ y_{n+\frac{1}{3}} \end{bmatrix} \qquad f(y) = \begin{bmatrix} f_n \\ f_{n+\frac{1}{9}} \\ f_{n+\frac{2}{9}} \\ f_{n+\frac{1}{3}} \end{bmatrix} \qquad Y_{i-1} = \begin{bmatrix} y_{n+\frac{1}{9}} \\ y_n \end{bmatrix} \qquad Y_{i+1} = \begin{bmatrix} y_{n+\frac{1}{9}} \\ y_{n+\frac{1}{3}} \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.9}$$

The entries of the matrices A, B, U, V, M and I are substituted into the equations (4.5) and (4.6) to get stability polynomial of the one-third step method which is then plotted in Matlab (R2012a) environment to generate the needed absolute stability region of our new methods as we can see in the figure below.

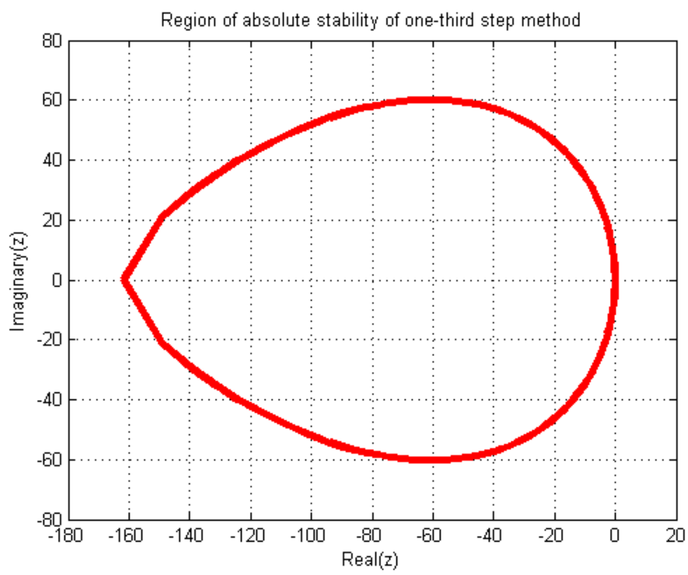

Fig. 1. Region of absolute stability of our new methods

From the figure above, it can be seen very clearly that our new method is A-Stable and the plot covers much region of the complex plane $z\epsilon\mathbf{C}^n$.

4.4 Consistency of the new method

According to Areo and Omojola [21], the linear hybrid multistep method is said to be consistent if all the following four conditions are satisfied

i the order of the method must be greater than or equal to one i.e. $(p \geq 1)$.

ii $\sum_{j=0}^{k} \alpha_j = 0$, where α_j's are the coefficients of the first characteristics polynomials $\rho(r)$

iii $\rho(r) = \rho'(r) = 0$ for $r = 1$

iv $\rho''(r) = 2!\sigma(r)$ for $r = 1$

where, $\rho(r)$ and $\sigma(r)$ are the first and second characteristics polynomials of our method. Following Lambert [22], the first condition is a sufficient condition for the associated block method to be consistent. Our method is order $p = 4 \geq 1$. Therefore the block is consistent.

4.5 Convergence of the new method

According to Jator [3], the two sufficient conditions for a linear hybrid multistep methods to be convergent are to be zero-stable and consistent. Since the two conditions are satisfied, we conclude that the new developed method converges.

5 Numerical Examples

In this section, we deal with the implementation of the block method in solving initial value problems (IVP) of second order ordinary differential equations. The method is coded in Matlab (R2012a) version environment using window 8.1 as an operating system. The new developed method is tested on some problems to determine the performance of the new proposed schemes and our solutions are compared with the results of other scholars in the literature. The following problems are chosen as a test problems.

Problem 1: We consider a test problem which also answered by Badmus [23].

$$y'' = 3y' + 8e^{2x} \text{ with initial condition } y'(0) = 1, y(0) = 1, h = 0.005$$

Exact solution:

$$y(x) = -4e^{2x} + 3e^{3x} + 2$$

Table 1. This table shows the results and comparison of test problem 1

X-value	Exact Result	Computed Result	Error in our Method	Error in [23] ($k = 3$)]	Time
0.0050	1.00513852551049	1.00513852427561	1.2349e-09	1.5800E-07	0.0230
0.0100	1.01055824175353	1.01055823906303	2.6905e-09	3.1760E-06	0.0306
0.0150	1.01626544391208	1.01626543953824	4.3738e-09	1.2941E-05	0.0389
0.0200	1.02226654286653	1.02226653657447	6.2921e-09	1.9323E-05	0.0743
0.0250	1.02856806714980	1.02856805818008	8.9697e-09	4.0181E-05	0.0822
0.0300	1.03517666493419	1.03517665407164	1.0863e-08	2.2075E-05	0.0923
0.0400	1.04934228403829	1.04934226757564	1.6463e-08	8.9916E-05	0.1206

Problem 2: We also consider test problem which was also solved by Badmus [23].

$$y'' = -\frac{6y'}{x} - \frac{4y}{x^2} \text{ with initial condition } y'(1) = 1, y(1) = 1, h = \frac{0.1}{32}, x > 0$$

Exact solution:

$$\frac{5}{3x} - \frac{2}{3x^4}, x > 0.$$

Table 2. This table shows the results and comparison of test problem 2

X-value	Exact Result	Computed Result	Error in our Method	Error in [23] ($k = 3$)	Time
1.003125	1.00307652585770	1.00307652589082	3.3125e-11	8.3000E-08	0.0254
1.006250	1.00605750308352	1.00605750323188	1.4837e-10	1.6000E-06	0.0346
1.009375	1.00894499508884	1.00894499542795	3.3911e-10	6.6380E-06	0.0437
1.012500	1.01174101816799	1.01174101876995	6.0196e-10	9.4910E-06	0.0532
1.015625	1.01444754268642	1.01444754362007	9.3365e-10	1.9535E-06	0.0617
1.018750	1.01706649423567	1.01706649556671	1.3310e-09	9.4160E-06	0.0712
1.021875	1.01959975475629	1.01959975654736	1.7911e-09	4.6505E-05	0.0812
1.025000	1.02204916362943	1.02204916594028	2.3108e-09	4.7122E-05	0.0908
1.028125	1.02441651873841	1.02441652162594	2.8875e-09	1.86926E-04	0.1011
1.031250	1.02670357750081	1.02670358101925	3.5184e-09	4.43321E-04	0.1103

Problem 3: We consider a specially oscillatory test problem which was also solved by Awoyemi et al. [24].

$$y'' = -\lambda y \text{ we take } , \lambda = 2, \text{ with initial condition } y'(0) = 2, y(0) = 1, h = 0.01$$

Exact solution:

$$y(x) = \cos 2x + \sin 2x.$$

Table 3. This table shows the results and comparison of test problem 3

X-value	Exact Result	Computed Result	Error in our Method	Error in [24]	Time
0.0100	1.01979867335991	1.01979867335983	8.4599e-14	2.6577E-11	0.0275
0.0200	1.03918944084761	1.03918944084726	3.4861e-13	8.4761E-10	0.0368
0.0300	1.05816454641465	1.05816454641386	7.8870e-13	6.4146E-09	0.0460
0.0400	1.07671640027179	1.07671640027039	1.4004e-12	6.7071E-09	0.0561
0.0500	1.09483758192485	1.09483758192267	2.1791e-12	7.1209E-09	0.0658
0.0600	1.11252084314279	1.11252084313966	3.1208e-12	7.6530E-09	0.0758
0.0700	1.12975911085687	1.12975911085265	4.2211e-12	8.3601E-09	0.0864
0.0800	1.14654548998987	1.14654548998440	5.4761e-12	9.0592E-09	0.0959
0.0900	1.16287326621395	1.16287326620707	6.8801e-12	9.9268E-09	0.1059
0.1000	1.17873590863630	1.17873590862787	8.4293e-12	1.0899E-08	0.1161

Problem 4: We consider a test problem which also answered by Anake et al. [12].

$$y'' = \frac{2y'}{x} + xe^x - y\left(1 + \frac{2}{x^2}\right) \text{ with initial condition } y'\left(\frac{\pi}{2}\right) = \frac{\pi}{4}\left(8 + e^{\frac{\pi}{2}}\right), y\left(\frac{\pi}{2}\right) = 4 - \pi + \frac{1}{4}\left(e^{\frac{\pi}{2}}\right)(\pi + 2),$$

$h = 0.003125$

Exact solution:

$$y(x) = 2x\cos x + 4x\sin x + \frac{1}{2}xe^x.$$

Table 4. This table shows the results and comparison of test problem 4

X-value	Exact Result	Computed Result	Error in our Method	Error in [12]	Time
1.7000	10.95785118097658	10.95785118024888	7.2770e-10	0.67798365E-06	0.3782
1.8000	11.63820762976944	11.63820762677736	2.9921e-09	0.77776457E-06	0.6323
1.9000	12.31472912025427	12.31472911259183	7.6624e-09	0.83164688E-06	0.9318
2.0000	12.99859200531184	12.99859198969727	1.5615e-08	0.81943241E-06	1.3239
2.1000	13.70481572693061	13.70481569919164	2.7739e-08	0.72051040E-06	1.6102
2.2000	14.45259109075712	14.45259104583181	4.4925e-08	0.51423437E-06	1.9050
2.3000	15.26561176327884	15.26561169523377	6.8045e-08	0.18028640E-06	2.3457
2.4000	16.17241142639471	16.17241132845798	9.7937e-08	0.30097603E-06	2.6602
2.5000	17.20670978769539	17.20670965230368	1.3539e-07	0.94819480E-06	3.0036
2.6000	18.40777146833077	18.40777128718930	1.8114e-07	0.17787125E-06	3.3458

6 Discussion of the Results

In this paper, we have used the procedure of collocation and interpolation to develop a uniform fourth order continuous one-third hybrid block method for the direct solutions of second order initial value problems of ordinary differential equations. In the Table 3 and Table 4, it is shown that our new method is more accurate than the methods proposed by Awoyemi, et al. [24] and Anake, et al. [12] which are of the same order four. It has been seen from Table 1 and Table 2 that our new method yield better results than the results presented by Badmus [23], despite the high order of his method, our new block method of order four are more efficient and accurate than his method of order eight.

7 Conclusion

We have proposed a new block method that give solutions to second order initial value problems directly without reducing to a system of first order ordinary differential equation. We used our new developed method to solve some numerical problems and the results obtained were significantly better when compared with those in the tables 1, 2, 3 and 4. Also, by implementing and running code for our new developed block method in Matlab environment verified that our method work very fast, since the running times for the solved problems are less than four seconds. We conclude that the new method gives better approximate solutions than some existing methods.

Competing Interests

Authors have declared that no competing interests exist.

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