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On Minimal Non-elementary Groups and Proper NF-groups

Yıldız Aydın[1*] and Ali Pancar[1]

[1] *Department of Mathematics, Faculty of Art and Science, Ondokuz Mayis University, Samsun, Turkey.*

Authors' contributions

This work was carried out in collaboration between both authors. Author YA designed the study, wrote the first draft of the manuscript and managed literature searches. Author AP read and fixed the manuscript. Both authors read and approved the final manuscript.

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Abstract

Relations between a subgroup having F-supplement and a minimal non-elementary group are investigated and as a conclusion some conditions are given for a minimal non-elementary group to be an F-group. Then proper NF-groups which are also F-groups are studied.

Keywords: Minimal non-elementary group; proper NF-group.

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1 Introduction

In a group G, the notation we used for Frattini subgroup is $Frat(G)$, for the center of the group $Z(G)$. A complement of a subgroup H of G is a subgroup S of G such that $G = HS$ and $H \cap S = 1$.

Corresponding author: E-mail: yildiz.aydin@omu.edu.tr;

An $nCS-group$ is a group in which every proper normal subgroup has a characteristic supplement in G. A group is called elementary by [1] if every subgroup of the group has trivial Frattini subgroup. In [2] Tani Corsi called group a $minimal\ non-elementary\ group$, if every proper subgroup has trivial Frattini Subgroup but the group itself has nontrivial Frattini subgroup. She also called group a $proper\ NF-group$, if every subnormal subgroup has trivial Frattini subgroup but the group itself has nontrivial Frattini subgroup. Kirtland studied minimal non-elementary groups and proper NF-groups in [3]. We call a group G is an $F-group$ if for every subgroup H of G, there exists a subgroup S of G such that $G = HS$ and $H \cap S \leq Frat(S)$ in [4]. Here the subgroup S is called an $F-supplement$ of H in G. We obtained some useful conclusions with this definition by theorems and corollaries in [5]. In [6] Lennox studied the groups in which Frattini factor of the group is finite and concluded that finitely generated groups which have finite Frattini factor are finite. The motivation of this study is to find relations between F-groups, minimal non-elementary groups and proper NF-groups. Firstly in Proposition 1 we concluded that if a group satisfies the both properties of being an F-group and minimal non-elementary group then every proper subgroup has a complement in the group. Then in Theorem 2, it is shown that for minimal non-elementary groups normal F-supplement is transitive and in the light of Theorem 2 in Corollary 3 we had the group is an F-group with extra conditions. In Proposition 4 and Corollary 5 proper NF-groups and F-groups are discussed. Finally in Theorem 6 and Corollary 7 we have shown that in proper NF-groups if a normal subgroup has the group as F-supplement then the normal subgroup is elementary abelian. All notions which are used here are taken from [7].

2 Basic Results and Definitions

Proposition 1. *Let G be a minimal non-elementary F-group. For any proper nontrivial subgroup H of G, if G is not F-supplement of H then H has a complement in G.*

Proof. Since G is an F-group for every $1 \neq H < G$, there exists a subgroup S of G such that $G = HS$ and $H \cap S < Frat(S)$. By hypothesis $S \neq G$ and since G is minimal non-elementary $Frat(S) = 1$. Hence $H \cap S < Frat(S) = 1$ and S is a complement of H in G. □

Theorem 2. *Let G be a minimal non-elementary group , $M < N < K$ be sequence of proper subgroups M, N and $K \leq G$. If M has a normal F-supplement in N and N has an F-supplement in K, then M has an F-supplement in K.*

Proof. If $M = 1$, K is a complement of M in K. So we can take $M \neq 1$. By hypothesis, M has a normal $F-supplement$ in N, say S. Then $N = MS$ and $M \cap S = 1$ by Proposition 1. Now if $K = G$, then N has an $F-supplement$ in G. So $K = G = NT$ and $N \cap T \leq Frat(T)$ for some $T \leq G$. If T is proper then T is a complement of N in G. In this case we have $K = G = NT = (MS)T = M(ST)$ since S is a normal subgroup of G. Now we will show that $M \cap ST = 1$. Let $a \in M \cap ST$. Since $a \in ST$ we write $a = st$ for some $s \in S$ and $t \in T$. $a = st \Rightarrow t = s^{-1}a \in SM = MS = N$ and hence $t \in N \cap T = 1$. Therefore, we have $a = s \in M \cap S = 1 \Rightarrow a = 1$. So ST is a complement of M in K. If $T = G$ then $G = NG$ and $N = N \cap G \leq Frat(G)$. Now, $G = NG = MSG = MG$ and $M \cap G = M < N < Frat(G) \Rightarrow M \cap G \leq Frat(G)$, hence G is an $F-supplement$ of M in $G = K$.

If K is proper subgroup of G then N has a complement, say T, in K and ST is a complement of M in K in similar way as mentioned in previous paragraph. □

Corollary 3. *Let G be a minimal non-elementary group, $M < N < K$ be sequence of proper subgroups M, N and $K \leq G$. If M has a normal F-supplement in N and N has a normal F-supplement in K, then G* is an $F-group$.

Proof. Let H be a subgroup of G. If $H = 1$ or $H = G$ then H has an $F - supplement$ in G. So we can assume $H \neq 1$ and H is proper subgroup of G. Let M be a maximal subgroup of G such that $H \leq M < G$. Now H has a normal $F - supplement$ in M and M has a normal $F - supplement$ in G by hypothesis. Finally, H has an $F - supplement$ in G, by Theorem 2. If G has no maximal subgroup then $Frat(G) = G$ and G is $F - supplement$ of H in G. Hence G is an $F - group$. □

Proposition 4. *Let G be a proper $NF - group$. If G is also an $F - group$ then 1 and G are only subgroups which have normal $F - supplement$.*

Proof. Since G is an $F - group$, every subgroup of G has an $F - supplement$ in G. 1 has G as $F - supplement$ and G has 1 as $F - supplement$. For an $1 \neq H < G$, assume that H has a normal $F - supplement$ in G. If N is a normal $F - supplement$ of H in G then $G = NH$ and $N \cap H \leq Frat(N)$ =1 since G is a proper $NF - group$. It means that G has a direct decomposition, which is a contradiction by [3]. □

Corollary 5. *Let G be a proper $NF - group$. If G is not a simple $F - group$ then G can not be an $nCS - group$.*

Proof. By Proposition 4 any normal subgroup N of G has normal $F - supplement$. So any of the normal subgroups has characteristic supplement. Hence G can not be an $nCS - group$. □

Theorem 6. *Let G be a finite proper $NF - group$. If G is the only $F - supplement$ of proper normal subgroup N of G then N is elementary abelian .*

Proof. Let G be a finite proper $NF - group$, N is a proper normal subgroup of G and G is $F - supplement$ of N in G. We know that G is the only $F - supplement$ of N by [4]. So we can write $G = NG$ and $N = N \cap G \leq Frat(G)$. Since $Frat(G)$ is elementary abelian by [3], then N is elementary abelian. □

Corollary 7. *Let G be a finite nonabelian proper $NF-group$. Then G is the only $F-supplement$ of $Z(G)$. In this case $Z(G)$ is elementary abelian.*

Proof. $Z(G) \cap G = Z(G) \leq Frat(G)$ by [3]. So G is the only $F - supplement$ of $Z(G)$. Finally, if G is finite then $Z(G)$ is elementary abelian by Theorem 6. □

3 Conclusion

In this study relations between minimal non-elementary groups, proper NF-groups and F-groups which we studied in our former work.

Competing Interests

Authors have declared that no competing interests exist.

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