British Journal of Mathematics & Computer Science
15(4): 1-11, 2016, Article no.BJMCS.24725
ISSN: 2231-0851

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Harmonic Solution of a Weakly Non-linear Second Order Differential Equation Governed the Motion of a TM-AFM Cantilever

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Authors' contributions

This work was carried out in collaboration between all authors. Since the birth of the idea through its development all the way to its completion. All the authors worked in co-operation to create the work featured in this paper. All the authors read and approved the final manuscript.

Article Information

DOI: 10.9734/BJMCS/2016/24725
Editor(s):
(1) Xiaodi Li, School of Mathematical Sciences, Shandong Normal University, Ji'nan, 250014, Shandong, P. R. China.
Reviewers:
(1) Teodoro Lara, University of Los Andes, Venezuela.
(2) Anonymous, University of Hacettepe, Ankara, Turkey.
Complete Peer review History: http://sciencedomain.org/review-history/13870

Received: 30th January 2016
Accepted: 9th March 2016
Published: 26th March 2016

Original Research Article

Abstract

The harmonic solution of a weakly non-linear second order differential equation governed the dynamic behavior of a micro cantilever based on TM (Tapping mode) AFM (Atomic force microscope) is investigated analytically by applying the method of multiple scales (MMS). The modulation equations of the amplitude and the phase are obtained, steady state solutions, frequency response equation, the peak amplitude with its location and the approximate analytical expression are determined. The stability of the steady state solutions is calculated. Numerical solutions of the frequency response equation and its stability condition are carried out for different values of the parameters in the equation. Results are presented in a group of figures. Finally discussion and conclusion are given.

Keywords: Micro-electro-mechanical system (MEMS); atomic force microscopy (AFM); differential equation; harmonic solution; multiple scales method.

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2010 Mathematics Subject Classification: 34K13, 34K27.

1 Introduction

Nowadays, the atomic force microscope (AFM) has become a useful tool for direct measurements of intermolecular forces with atomic precision. AFM has been developed to a nearly ubiquitous tool for studying Physics, chemistry, biology, medicine and engineering at the nano-scale [1]. AFM could significantly impact many fabrication and manufacturing processes due to its advantages such as 3D topography of nano-fabrication and metrology for micro-electro-mechanical systems (MEMS) and it permits the imaging and probing of nano-mechanical properties as biopolymers and viruses under physiological (liquid environment) conditions. The ability of AFM to measure forces in the nano-Newton range under physiological conditions makes it a very attractive tool for studying many biological interactions and intermolecular force governed phenomena [2–6]. Micro-scale structural such as micro beams, micro plates and microbars are widely used in MEMS and AFM. The experimental observations have indicated that the mechanical behavior of the micro/nano structures is size-dependent. Since the classical continuum mechanics is incapable of capturing the size effect and consequently unable to predict and interpret the size-dependent static and the oscillation behavior observed in micro scale structure [7–10].

Most of classical dynamical systems and non-classical dynamical systems (Micro/Nano mechanical systems), its mathematically study leads to a nonlinear second ordinary differential equations or a set of a nonlinear coupled second ordinary differential equations. It is clear that the solutions of the differential equations other than periodic solutions exist. However, the literature on the subject is almost entirely devoted to the different types of periodic solutions (harmonic, sub, super, sub-super and super-sub harmonic solutions).

In the past three decades, there has been great interest to obtain the different types of periodic solutions of second order nonlinear ordinary differential equations. Elnaggar [11] investigated a harmonic oscillation as solutions to physical systems governed by quasi-linear differential equation. Elnaggar and Ahmed introduced a Perturbation method for a set of quasi-linear oscillatory systems with variable natural frequency [12]. Elnaggar and El-Bassiouny [13] discussed the harmonic solution of self-excited two coupled second order systems to multi-frequency excitations. Shooshtari and Pasha Zanoosi [14] investigated the harmonic solution of second order weakly non-linear differential equation that represent the vibration of a mass grounded system which includes two linear and nonlinear springs in series by using multiple scales method [15, 16] and some analytical relations have been obtained for natural frequency of oscillations and the effects of different parameters on the frequency response have been investigated. The harmonic solution of a forced single degree of freedom (SDOF) nonlinear system was represented by Elnaggar *et al.* [17] by using the method of multiple scales. Mazaheri *et al* [18] studied the nonlinear oscillation of a pendulum wrapping and unwrapping on two cylindrical bases. Pilkee Kim *et al.* [19] investigated the harmonic solution of the dynamic behaviors of a nonliner cantilever beam with tip mass subject to an axial force and electrostatic excitation. Hanna Cho *et al.* [20] studied analytically the micro-mechanical cantilever system integrated with geometric nonlinearity to determine its dynamic behavior and used The method of multiple scales to find the dynamic response of the system at the fundamental mode resonance of the micro cantilever. Kirrou and Belhaq [21] investigated analytically the harmonic solution of Contact stiffness modulation in contact-mode atomic force microscopy. Elnaggar *et al.* [22] studied harmonic and sub-harmonic solutions of a van der Pol equation subjected to weakly non-linear parametric and forcing excitations. The non-linear governing equations of the non-linear forced vibration of strain gradient micro beams are solved analytically by Vatankhah *et al.* [23] using the perturbation techniques. Elnaggar and Khalil [24] investigated harmonic solution for non-linear (SDOF) system with two distinct time-delays under an external excitation. Harmonic solution of single degree of freedom system which describe non-contact mode AFM and the stability of solution are discussed by Kirrou and Belhaq [25]. Elnaggar *et al.* [26] discussed the electrostatically actuated MEMS resonant sensors which represented by a modified Duffing-Van der Pol equation subjected to weakly non-linear parametric and external excitations. The harmonic solution of weakly nonlinear second order differential equation that represents a doubly clamped micro beam based resonator driven by two electrodes by Han *et al.* [27]. Wen-Ming Zhang *et al.* [28] studied the dynamic behavior of a micro cantilever based TM-AFM with squeeze film damping effects using numerical simulation.

The interest of this work is to study the harmonic solution (i.e. periodic solution with period equal to the period of external excitation) of a weakly non-linear second order differential equation governed the motion of micro cantilever based TM-AFMs with squeeze film damping by using the perturbation technique (Method of multiple scales). The modulation equations of the amplitude and the phase are determined. Steady state solutions and its stability are given. Peak amplitude and its localization are determined. Numerical solutions for the frequency response equation and the stability conditions are carried out. Results are presented in group of figures in which solid (dashed) curves represented stable (unstable) harmonic solutions. Finally, discussion and conclusion are given.

2 Formulation of the Problem and Perturbation Analysis

Consider the following non-linear second order differential equation

$$y'' + \zeta y' + y + \beta y^3 = -\frac{d}{(\alpha+y)^2} + \frac{d\Sigma^6}{30(\alpha+y)^8} + \epsilon\left(f\text{Cos}\Omega t - \frac{\eta}{(\alpha+y)^3}y'\right) \tag{2.1}$$

where $\alpha,\ \beta,\ d,\ \Sigma,\ \zeta,\eta,\ \Omega$, f and $\epsilon << 1$ are constants. Eq.(2.1) represents the mathematical model of the dynamic behavior of a micro cantilever based TM-AFM with squeeze film damping effects [28].

By using Taylor expansion on the right side of Eq.(2.1), keeping only three terms of its result and for applying a perturbation technique, we must take the non-linear terms of order ϵ then we get the following weakly non-linear second order differential equation:

$$y'' + \omega_0^2 y + \epsilon(2\mu y' - \alpha_3 y^2 + \beta y^3 - \alpha_5 y y' + \alpha_6 y^2 y') = \alpha_1 + \epsilon f\text{Cos}\Omega t \tag{2.2}$$

where $\omega_0^2 = 1-\alpha_2, 2\mu = \zeta + \alpha_4 = \zeta + \frac{\eta}{\alpha^3}, \alpha_1 = \frac{d\Sigma^6}{30\alpha^8} - \frac{d}{\alpha^2}, \alpha_2 = \frac{2d}{\alpha^3} - \frac{4d\Sigma^6}{15\alpha^9}, \alpha_3 = \frac{6d\Sigma^6}{5\alpha^{10}} - \frac{3d}{\alpha^4}$,
$\alpha_4 = \frac{\eta}{\alpha^3}, \alpha_5 = \frac{3\eta}{\alpha^4}$ and $\alpha_6 = \frac{6\eta}{\alpha^5}$.

An approximate solution of Eq.(2.2) can be obtained by a number of perturbation techniques (Nayfeh [15, 16]). In this paper we use the method of multiple scales (MMS). Acording to MMS, the scaled times T_n can be introduced as:

$$T_n = \epsilon^n t, \qquad n = 0, 1, 2, ... \tag{2.3}$$

Differentiation with respect to the dimensionless time t we obtain

$$\frac{d}{dt} = D_0 + \epsilon D_1 + ... \qquad \& \qquad \frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + ... \tag{2.4}$$

where $D_n = \frac{\partial}{\partial T_n}$. we assume a two scale expansion of the solution of Eq. (2.2) in the form

$$y(t;\epsilon) = y_0(T_0, T_1) + \epsilon y_1(T_0, T_1) + ..., \tag{2.5}$$

Substituting Eqs. (2.4) and (2.5) into Eq. (2.2) then equating the coefficients of like powers of ϵ to zero, we obtain a set of linear partial differential equations

$$D_0^2 y_0 + \omega_0^2 y_0 = \alpha_1 \tag{2.6}$$

$$D_0^2 y_1 + \omega_0^2 y_1 = f\mathrm{Cos}\Omega t - 2\mu D_0 y_0 - 2D_1 D_0 y_0 - \beta y_0^3 + \alpha_3 y_0^2 + \alpha_5 y_0 D_0 y_0 - \alpha_6 y_0^2 D_0 y_0 \tag{2.7}$$

By solving the equation (2.6) for $y_0(T_0, T_1)$, we have

$$y_0(T_0, T_1) = A(T_1) e^{i\omega_0 T_0} + \bar{A}(T_1) e^{-i\omega_0 T_0} + \Lambda \tag{2.8}$$

where $i^2 = 1$, $\bar{A}$ is the complex conjugate of A and $\Lambda = \frac{\alpha_1}{\omega_0^2}$ substituting Eq. (2.8) into Eq. (2.7), we get

$$\begin{aligned} D_0{}^2 y_1 + \omega_0^2 y_1 = &- e^{iT_0\omega_0}\big(A\left(i\omega_0\left(\alpha_6\left(A\bar{A} + \Lambda^2\right) - \alpha_5\Lambda + 2\mu\right) + 3A\beta\bar{A} - 2\alpha_3\Lambda + 3\beta\Lambda^2\right) \\ &+ 2i\omega_0 A'\big) + \frac{1}{2} f e^{iT_0\Omega} + 2A\bar{A}\left(\alpha_3 - 3\beta\Lambda\right) + \Lambda^2\left(\alpha_3 - \beta\Lambda\right) + NST. + c.c. \end{aligned} \tag{2.9}$$

where $NST.$ denotes the terms does not produce secular terms and *c.c.* denotes the complex conjugate.

3 Harmonic Solution

To analyze the harmonic solution, it is assumed that the frequency of the external excitation Ω and the natural frequency ω_0 of the corresponding linear system are closed to each other, i.e. $\Omega \approx \omega_0$. Hence, this previous expression can be written as:

$$\Omega = \omega_0 + \epsilon\sigma \tag{3.1}$$

where σ is a detuning parameter. Then the excitation can be expressed in terms of T_0 and T_1 as

$$f\cos\Omega t = f\cos\left(\omega_0 T_0 + \sigma \mathrm{T}_1\right) \tag{3.2}$$

And so by eliminating the secular terms(coefficient of $e^{i\omega_0 T_0}$) from the Eq . (2.9) yields

$$A\left(-i\omega_0\left(\alpha_6\left(A\bar{A} + \Lambda^2\right) - \alpha_5\Lambda + 2\mu\right) - 3A\beta\bar{A} + 2\alpha_3\Lambda\right) - 2i\omega_0 A' - 3A\beta\Lambda^2 + \frac{1}{2} f e^{i\varepsilon\sigma T_0} = 0 \tag{3.3}$$

Eq.(3.3) is a differential equation in complex form. In order to solve it,$A(T_1)$ can be expressed in polar form as:

$$A = \frac{1}{2} a(T_1) e^{i\beta_1(T_1)} \tag{3.4}$$

Where a and β_1 are real functions of T_1. By substituting Eq. (3.4) into Eq. (3.3) and separating the real and imaginary parts respectively, we obtain a set of autonomous differential equations that govern the amplitude $a(T_1)$ and the phase $\gamma(T_1)$

$$\begin{aligned} a' &= \tfrac{1}{8\omega_0}\left\{\omega_0\left(a\left(-\alpha_6\left(a^2 + 4\Lambda^2\right) + 4\Lambda\alpha_5 - 8\mu\right)\right) + 4f\mathrm{Sin}\gamma\right\} \\ a\gamma' &= \tfrac{1}{8\omega_0}\left\{a\left(-3a^2\beta + 4\Lambda\left(2\alpha_3 - 3\beta\Lambda^2\right) + 8\sigma\omega_0\right) + 4f\mathrm{Cos}\gamma\right\} \end{aligned} \tag{3.5}$$

where $\gamma = \sigma \mathrm{T}_1 - \beta_1$ and system (3.5) is known by the modulation equations of the amplitude and the phase.

4 Steady State Solution

Applying the steady state conditions, i.e., $a' = \gamma' = 0$, we have a set of algebraic equations for amplitude a and phase γ of the steady state harmonic solution.

$$\begin{aligned} a\left(\alpha_6\left(a^2+4\Lambda^2\right)-4\Lambda\alpha_5+8\mu\right)\omega_0 &= 4f\mathrm{Sin}\gamma \\ a\left(3a^2\beta-4\left(\Lambda\left(2\alpha_3-3\beta\Lambda^2\right)\right)-8\sigma\omega_0\right) &= 4f\mathrm{Cos}\gamma \end{aligned} \tag{4.1}$$

Squaring both equations in system (4.1) and adding the results, we get the frequency response equation in the form

$$\begin{aligned} &a^2((-3\beta(a^2+4\Lambda^2)+8\Lambda\alpha_3))^2-16a^2\sigma(3\beta(a^2+4\Lambda^2)-8\Lambda\alpha_3)\omega_0+a^2(64\sigma^2+(8\mu-4\Lambda\alpha_5 \\ &+a^2\alpha_6+4\Lambda^2\alpha_6)^2)\omega_0^2-16f^2=0 \end{aligned} \tag{4.2}$$

solving equation (4.2)for σ, we obtain

$$\begin{aligned} \sigma = &\frac{1}{8a^2\omega_0^2}\{a^2\omega_0\left(3\beta\left(a^2+4\Lambda^2\right)-8\alpha_3\Lambda\right) \\ &\pm\sqrt{a^2\omega_0^2\left(16f^2-a^2\omega_0^2\left(\alpha_6a^2+4\alpha_6\Lambda^2-4\alpha_5\Lambda+8\mu\right)^2\right)}\} \end{aligned} \tag{4.3}$$

The peak amplitude would be verifying the following equation

$$16f^2-\omega_0^2a_p^2\left(4\alpha_6\Lambda^2-4\alpha_5\Lambda+\alpha_6a_p^2+8\mu\right)^2=0 \tag{4.4}$$

Then the corresponding value of σ is given from

$$\sigma_p=\frac{3\beta\left(a_p^2+4\Lambda^2\right)-8\alpha_3\Lambda}{8\omega_0} \tag{4.5}$$

From Eqs. (4.4) and (4.5) we can conclude that

- The peak amplitude value doesnt affect with change β.
- The peak amplitude location is affected by changing β, α_3, Λ and ω_0.

Therefore, the approximate analytical expression of the harmonic solution is

$$y = a\cos(\Omega t-\gamma)+\frac{\alpha_1}{\omega_0^2}+O(\epsilon) \tag{4.6}$$

Where a and γ are the amplitude and phase of the steady state solutions.

The stability of the steady state solutions can be examined by introducing a small perturbation to solutions obtained from system (3.5) i. e. by substituting

$$a = a_0 + a_1 \tag{4.7}$$

$$\gamma = \gamma_0 + \gamma_1 \tag{4.8}$$

Where a_0 and γ_0 represent the steady state solution a_1 and γ_1 represent small perturbations. Substituting Eqs.(4.7) and (4.8) into system (3.5) by using the steady state condition and keeping linear terms, one obtains

$$a_1' = \frac{1}{8\omega_0}\{-\omega_0\left(\alpha_6\left(3a_0^2+4\Lambda^2\right)-4\alpha_5\Lambda+8\mu\right)a_1+a_0\left(3\beta\left(a_0^2+4\Lambda^2\right)-8\alpha_3\Lambda-8a_0\sigma\omega_0\right)\gamma_1\} \tag{4.9}$$

$$\gamma_1' = \frac{1}{8a_0\omega_0}\{\left(-9a_0^2\beta+8\alpha_3\Lambda-12\beta\Lambda^2+8\sigma\omega_0\right)a_1-a_0\omega_0\left(\alpha_6\left(a_0^2+4\Lambda^2\right)-4\alpha_5\Lambda+8\mu\right)\gamma_1\} \tag{4.10}$$

Substituting $a_1 = \Gamma_1 e^{\theta T_1}$ and $\gamma_1 = \Gamma_2 e^{\theta T_1}$ into equations (4.9) and (4.10). We get

$$\left(4\Lambda\left(3\beta\Lambda - 2\alpha_3\right) + 9a_0^2\beta - 8\sigma\omega_0\right)\Gamma_1 + a_0\omega_0\left(a_0^2\alpha_6 + 4\alpha_6\Lambda^2 - 4\alpha_5\Lambda + 8\theta + 8\mu\right)\Gamma_2 = 0 \tag{4.11}$$

$$\omega_0\left(3a_0^2\alpha_6 + 4\alpha_6\Lambda^2 - 4\alpha_5\Lambda + 8\theta + 8\mu\right)\Gamma_1 + \left(a_0\left(8\alpha_3\Lambda - 3\beta\left(a_0^2 + 4\Lambda^2\right)\right) + 8a_0\sigma\omega_0\right)\Gamma_2 = 0 \tag{4.12}$$

For the nontrivial solution the determinant of the coefficient matrix for Γ_1 and Γ_2 must vanish, which leads to a quadratic equation for the eigenvalue θ.

$$\begin{aligned}\theta = &-\frac{1}{4}\left(\alpha_6\left(a_0^2 + 2\Lambda^2\right) - 2\alpha_5\Lambda + 4\mu\right)\\ &\pm\frac{1}{8\omega_0^2}\sqrt{\omega_0^2\left(a_0^4\left(\alpha_6^2\omega_0^2 - 27\beta^2\right) + 48a_0^2\beta\left(2\alpha_3\Lambda - 3\beta\Lambda^2 + 2\sigma\omega_0\right) - 16\left(2\alpha_3\Lambda - 3\beta\Lambda^2 + 2\sigma\omega_0\right)^2\right)}\end{aligned} \tag{4.13}$$

The stability of the harmonic solution can be examined by evaluating the sign of the real part of the eigenvalues. Consequently, a solution is stable if and only if the real parts of both eigenvalues of equation (4.13) are less than zero.

5 Numerical Results and Discussion

In this section, the frequency response Eq. (16) and its stability condition (27) are solved numerically for different values of the parameters. Results are presented in a group of Figs. (1-7) which represent the relations between the amplitude of the periodic solutions a and the detuning parameter σ at different values of the parameters. In all figures, the results obtained by computer simulation of Eq. (16) are plotted where the solid lines refer to stable solutions and the dashed lines represent the unstable ones.

Fig.1 shows the variation of the amplitude of the steady state solutions for different values of α. This figure shows that, for small values of α, there is one stable solution. But by increasing the parameter , the response amplitude bent to the right which gives hardening behavior and has two branches (singlevalued curve and semioval) where the singlevalued curve has stable solutions and the semi-oval has stable and unstable solutions. Also, the response amplitude loses stability via saddle node bifurcation when α is increased. Since the existence of saddle node bifurcation leads to the unwanted jump phenomena, so it prefers to use small values for α.

Fig.2 illustrates the influence of the coefficient of cubic term β on the response curves. We note that for $\beta = 0.0023$, we have single stable symmetric solution. As β increases, the frequency response curves consist of two branches; the left one is stable and the right one has two parts one of them is stable and the other part is unstable. These curves are bent to the right; the bending leads to multi-valued solutions and hence jump phenomena for certain values of σ exists. Also, the peak amplitude is shifted to the right.

In Fig.3, for large values of η, we have one stable solution. By decreasing η, we have three solutions; two curves are stable and one unstable; jump phenomena for certain values of σ and the inclination towards the R.H.S.

From Fig.4, for small values of the excitation amplitude f, we have one symmetric stable solution. By increasing f, we have multi-valued solutions for a certain value of σ which two of them are stable solutions and one unstable. There exist Jump phenomena and the bend towards the R.H.S.

In Fig.5 indicates that for large values of Σ, we have one-valued solution. By decreasing Σ, we have multi-valued solutions for a certain value of σ consist of two stable solutions and one unstable. Jump phenomena and the bend to the R.H.S.

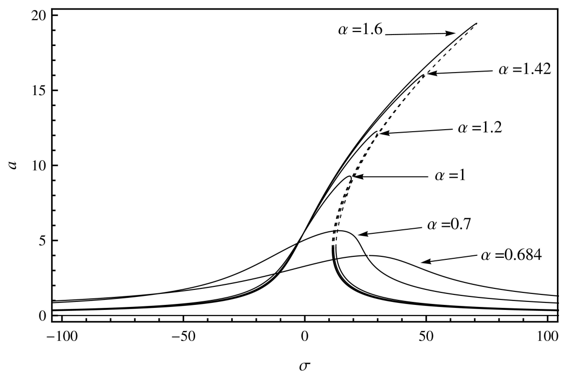

Fig. 1. The frequency response curves for the parameters
$\beta = 0.48, \eta = 0.0635, f = 65, \Sigma = 0.3, \zeta = 0.01, d = 4/27$ **and for different values of** α

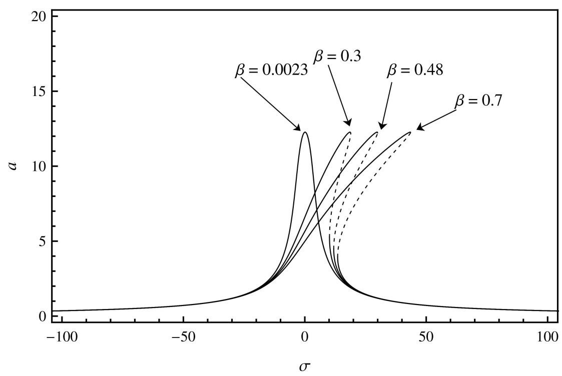

Fig. 2.The frequency response curves for the parameters
$\alpha = 1.2, \eta = 0.0635, f = 65, \Sigma = 0.3, \zeta = 0.01, d = 4/27,$ **and for different values of** β

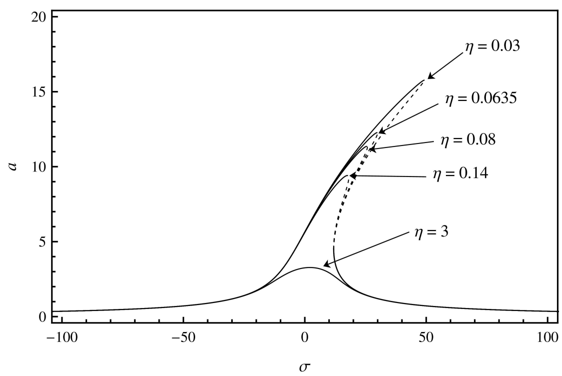

Fig. 3. The frequency response curves for the parameters
$\alpha = 1.2, \beta = 0.48, f = 65, \Sigma = 0.3, \zeta = 0.01, d = 4/27$ **and for different values of** η

In Fig.6, we observe that the frequency response for different values of ζ. For large values of ζ, we have single stable solution. By decreasing ζ, we have multi-valued solutions for a certain value of σ; two stable solutions and one unstable. Jump phenomena and the bend towards the R.H.S.

Fig.7 shows the effect of d on the frequency response indicating that by increasing d, we have multi-valued solutions for a certain value of σ, two stable solutions and one unstable solution and jump phenomena and the bend in the R.H.S.

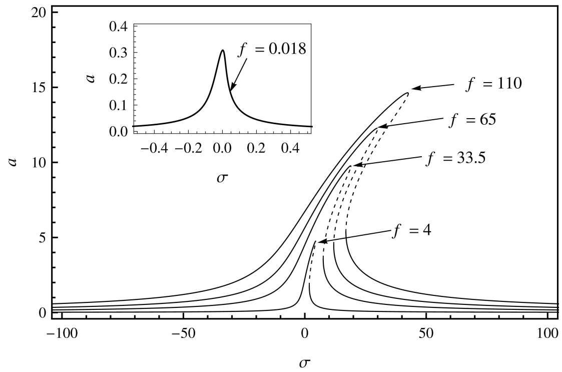

Fig. 4. The frequency response curves for the parameters $\alpha = 1.2, \beta = 0.48, \eta = 0.0635, \Sigma = 0.3, \zeta = 0.01, d = 4/27$ and for different values of f

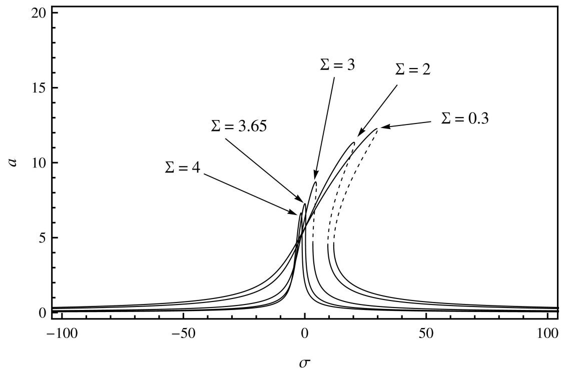

Fig. 5. The frequency response curves for the parameters $\alpha = 1.2, \beta = 0.48, \eta = 0.0635, f = 0.5, \zeta = 0.01, d = 4/27$ and for different values of Σ

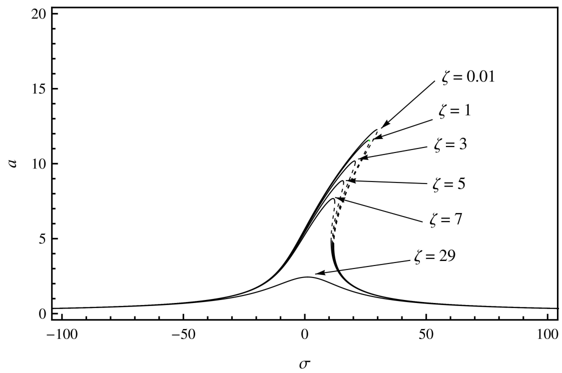

Fig. 6. The frequency response curves for the parameters $\alpha = 1.2, \beta = 0.48, \eta = 0.0635, , f = 65, \Sigma = 0.3, d = 4/27$ and for different values of ζ

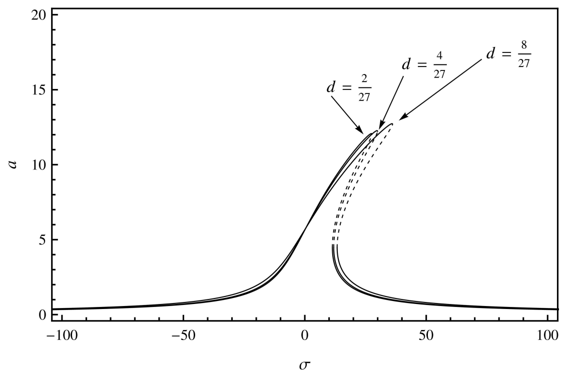

Fig. 7. The frequency response curves for the parameters $\alpha = 1.2, \beta = 0.48, \eta = 0.0635, f = 65, \Sigma = 0.3, \zeta = 0.01$ and for different values of d

6 Conclusion

In this paper, we have presented an analysis of harmonic solution for a weakly non-linear second order differential equation which governed the dynamic behavior of a micro cantilever based on TM (Tapping mode) AFM (Atomic force microscope) . The method of multiple scales is used to determine two first order ordinary differential equations which describe the modulation of the amplitude and the phase. Steady state solution and its stability are investigated. Peak amplitude and its localization are determined. Numerical solutions of the frequency response equation and the stability equation are carried out for different values of the parameters in the equation. It's known that the external excitation f and the squeeze damping η are important for the design of dynamic of cantilever in TM-AFM so, we investigated the model (1) to obtain f and η values which give the best results for that model ; the best result means obtaining stable solutions which haven't discontinues points then, we enhance the work of the system. Results are represented in group of figures in which solid curves (dashed) are denoted stable (unstable) solutions.

Competing Interests

Authors have declared that no competing interests exist.

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