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Invariant Subspace Method and Some Exact Solutions of Time Fractional Modified Kuramoto-Sivashinsky Equation

A. Ouhadan[1] and E. H. El Kinani[2,3*]

[1] *Centre Régional des Métiers de l'Education et de la Formation, Meknès, BP 255, Morocco.*
[2] *Université Moulay Ismaïl Ecole Nationale Supérieure des Arts et Métiers (ENSAM), Marjane 2, B.P. 15290, Meknès, Morocco.*
[3] *Department of Mathematical, Faculty of Sciences and Technics, Moulay Ismaïl University, Errachidia, BP 509, Morocco.*

Authors' contributions

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Abstract

In this paper, we construct the exact solutions of the modified nonlinear time fractional Kuramoto-Sivashinsky equation by suing the invariant subspace method. As a result, the obtained reduced system of nonlinear ordinary fractional equations is solved by the Laplace transform method and with using of some useful properties of Mittag-Leffler functions. Then, some exact solutions of the time fractional nonlinear studied equation are found.

Keywords: Invariant subspace method; caputo fractional derivative; time fractional modified kuramoto-sivashinsky equation; mittag-lefller functions.

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**Corresponding author: E-mail: elkinani_67@yahoo.com;*

1 Introduction

In the last decade, fractional calculus attracted a great interest of many researchers. The idea of fractional order derivative was started with half-order derivative as discussed in the literature by Leibniz and L'Hôpital. Next, it was extended to an arbitrary order derivative by Liouville, Riemann, Grünwald, Letnikov, Caputo etc. In addition, different approaches to define fractional derivatives are known [1], [2], [3], [4]. The study of fractional differential equations becomes of great interest, since for their widely applications including fluid flow, dynamical processes in self-similar and porous structures, electromagnetic waves, probability and statistics, viscoelasticity, signal processing, and so on [1], [4], [5].

The construction of particular exact solutions of fractional differential equations is not an easy task and it remains a relevant problem. This is the reason why a powerful methods for solving those fractional equations were recently developed in the literature, including Adomian's decomposition method [6], first integral method [7], homotopy perturbation method [8], Lie group theory method [9], [10], [11], [12], [13] and so on. Most recently, according to invariance principles, the invariant subspace method developed by V.A. Galaktionov and S.R. Svirshchevski [14] to study partial differential equations was extended by R.K. Gazizov and A.A. Kasatkin [15] to construct some particular exact solutions for time fractional differential equations.

Here we will use the invariant subspace method, this latter yields us with exact solutions of the time fractional modified Kuramoto-Sivashinsky equation in terms of the well known Mittag-Leffler functions. In the paper [15], the invariant subspace method and Lie group analysis are joined to solve the reduced fractional ordinary differential system and the original studied equation. In our case, the resolution of the reduced system is done by the Laplace transform method and by using of some remarkable properties of the well known Mittag-Leffler functions [16], [17], [18].

This paper is arranged as follows: In section 2, we recall some main results of fractional derivatives and integrals. Section 3, is devoted to describe the invariant subspace method. While in section 4, we use the described method to construct some exact solutions of the time fractional modified Kuramoto-Sivashinsky equation. In section 5 and in section 6, we extract some particular exact solutions corresponding to different values of parameters and we draw their 3-D surfaces. Finally, some remarks are in order.

2 Some Basic Results on Fractional Calculus

This section is devoted to recall briefly some definitions and basic results on fractional calculus. For more details and proofs of the results, we refer to [1], [2], [3], [4].

The Riemann-Liouville fractional integral is defined by:

$$J_t^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau)\, d\tau, \tag{2.1}$$

where $\gamma \in \mathbb{R}^+$, and

$$\Gamma(\gamma) = \int_0^{+\infty} x^{\gamma-1} e^{-x}\, dx, \tag{2.2}$$

is the Euler Gamma function.

By definition we have $J_t^0 f(t) = f(t)$ and it satisfies the stability property $J_t^{\gamma_1} J_t^{\gamma_2} f(t) = J_t^{\gamma_1+\gamma_2} f(t)$.

Before going on, recall that there are various contributions [1], [2], [3], [4] to define fractional derivatives. In this paper, we adopt the fractional derivative in the sense of Caputo [1], [2],[3], [4].

The Caputo definition is used not only because it makes easy the consideration of initial conditions but also because the derivative of a constant is equal to zero. In what follows, we recall some important results and properties of fractional Caputo derivatives and integrals. For more details see for example [4]. First, let us denote by $AC^n([0,t]), n \in \mathbb{N}$ the class of functions $f(x)$ which are continuously differentiable in $[0,t]$ up to order $(n-1)$ and with $f^{(n-1)} \in AC([0,t])$.

Theorem 2.1. *Let $n-1 < \alpha < n$, with $n \in \mathbb{N}$. If $f(x) \in AC^n([0,t])$, then the Caputo fractional derivative exists almost everywhere on $[0,t]$ and it is represented in the form:*

$$D_t^\alpha f(t) = J_t^{n-\alpha} D_t^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, \quad \alpha \neq n. \tag{2.3}$$

Then, the Caputo derivative (2.3) and the Riemann-Liouville integral (2.1)satisfy the following properties [4]:

$$\begin{aligned} D_t^\alpha J_t^\alpha f(t) &= f(t), \qquad \alpha > 0, && (2.4)\\ J_t^\alpha D_t^\alpha f(t) &= f(t) - \sum_{k=0}^{n-1} f^{(k)}(0)\frac{t^k}{k!}, \qquad \alpha > 0, \quad t > 0, && (2.5)\\ J_t^\alpha t^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}, \qquad \alpha > 0, \quad \gamma > -1, \quad t > 0, && (2.6)\\ D_t^\alpha t^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, \alpha > 0, \gamma \in]-1, 0[\cup]0, +\infty[,\, t > 0. && (2.7) \end{aligned}$$

Here, it is important to mention that the studied equation is a time fractional partial differential equation of order $0 < \alpha < 1$, so the integer n appearing in the relation (2.3) is equal to one. Consequently, the formulae (2.5) becomes:

$$J_t^\alpha D_t^\alpha f(t) = f(t) - f(0), \qquad t > 0. \tag{2.8}$$

3 Description of the Invariant Subspace Method

The aim of this section is to present some necessary and essentials results from invariant subspace theory. The invariant subspace method [14] was firstly used to construct particular exact solutions of evolutionary partial differential equations of the form:

$$\frac{\partial u}{\partial t} = F(u, u_{1x}, u_{2x}, \ldots, u_{kx}), \quad k \in \mathbb{N}, \tag{3.1}$$

where $u = u(t,x), u_{ix} = \frac{\partial^i u}{\partial x^i}$ is the i-th order derivative of u with respect to the space variable x and F is a nonlinear differential operator.

Recently, Gazizov and Kasatkin [15] showed that the invariant subspace method can be applied also to equations with time fractional derivative:

In fact, consider the time fractional partial differential equation of the form:

$$D_t^\alpha u(t,x) = F[u], \tag{3.2}$$

where $F[u] = F(u, u_{1x}, u_{2x}, \ldots, u_{kx})$ and D_t^α is the time fractional derivative in the sense of Caputo. The invariant subspace method is based on the following basic definitions and results [14], [15].

Let $f_1(x), \ldots, f_n(x)$ be an n linearly independent functions and W_n is the n-dimensional linear space namely $W_n = \langle f_1(x), \ldots, f_n(x)\rangle$. W_n is said to be invariant under the given operator $F[u]$ if $F[u] \in W_n$ whenever $u \in W_n$.

Proposition 3.1. *Let W_n be an invariant subspace of $F[u]$. A function $u(t,x) = \sum_{i=1}^{n} f_i(x)u_i(t)$ is a solution of equation (3.2) if and only if the expansion coefficients $u_i(t)$ satisfy the following system of fractional ordinary differential equations:*

$$\begin{cases} D_t^\alpha u_1 &= F_1(u_1,\dots,u_n), \\ D_t^\alpha u_2 &= F_2(u_1,\dots,u_n), \\ \quad\vdots & \qquad\vdots \\ D_t^\alpha u_n) &= F_n(u_1,\dots,u_n), \end{cases}$$

where $F_1,\dots,F_n$ are given by:

$$F(c_1f_1(x)+\dots+c_nf_n(x)) = F_1(c_1,\dots,c_n)f_1(x)+\dots+F_n(c_1,\dots,c_n)f_n(x). \tag{3.3}$$

Remark 3.1. A crucial question in the theory of invariant subspace method was how to get the corresponding invariant subspace of a given differential operator. This question is solved by the following proposition and for more details see [14].

Proposition 3.2. *Let $f_1(x),\dots,f_n(x)$ form the fundamental set of solutions of a linear n-th order ordinary differential equation*

$$L[u] = y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_{n-1}(x)y' + a_n(x)y = 0, \tag{3.4}$$

and $F[y] = F(x,y,y',\dots,y^{(k)})$ a given differential operator of order $k \le n-1$, then the subspace $W_n = \langle f_1(x),\dots,f_n(x)\rangle$ is invariant with respect to F if and only if:

$$L[F[y]] = 0, \tag{3.5}$$

whenever y satisfies equation (3.4).

Remark 3.2. Condition of invariance appearing in the above proposition is the invariance criterion for equation (3.4) with respect to the Lie-Bäcklund generator $V = F[y]\frac{\partial}{\partial y}$. This criterion shows us how the invariant subspace method is related to the techniques used in Lie symmetry analysis, see for more details [19], [20], [21], [22].

4 Exact Solution of the Fractional mKS Equation

In this section, we use the invariant subspace method to construct some exact solutions of the time fractional modified Kuramoto-Sivashinsky equation (mKS) which is given by:

$$D_t^\alpha(u) = -u_{4x} - u_{2x} + (1-\lambda)(u_x)^2 + \lambda(u_{xx})^2, \quad 0 < \alpha \le 1, \tag{4.1}$$

where $u = u(t,x)$ and $\lambda \in]0;1[$. In the case $\alpha = 1$, the (mKS) equation (4.1) is a model for the dynamics of a hyper-cooled melt [23], [24]. A more general class of such models was introduced and discussed in [25]. The (KS) was examined as a prototypical example of spatiotemporal chaos in one dimension and its was originally derived in the context of plasma instabilities [26].

Proposition 4.1. *For any $\lambda \in]0;1[$ the nonlinear operator $F[u]$ given by:*

$$F[u] = -u_{4x} - u_{2x} + (1-\lambda)(u_x)^2 + \lambda(u_{xx})^2, \tag{4.2}$$

admits $W_3 = \langle 1, \cos\gamma x, \sin\gamma x\rangle$ with $\gamma = \sqrt{\frac{1-\lambda}{\lambda}}$ as an invariant subspace.

Proof. For any function

$$h(t,x) = C_1 + C_2 \cos\gamma x + C_3 \sin\gamma x, \tag{4.3}$$

with $C_i = C_i(t)$ arbitrary functions, we get:

$$\begin{aligned} F[h] &= -\gamma^4 C_2 \cos\gamma x - \gamma^4 C_3 \sin\gamma x + \gamma^2 C_2 \cos\gamma x + \gamma^2 C_3 \sin\gamma x \\ &\quad +(1-\lambda)(\gamma^2 C_3^2 \cos^2\gamma x + \gamma^2 C_2^2 \sin^2\gamma x - 2\gamma^2 C_2 C_3 \cos\gamma x \sin\gamma x) \\ &\quad +\lambda(\gamma^4 C_2^2 \cos^2\gamma x + \gamma^4 C_3^2 \sin^2\gamma x + 2\gamma^4 C_2 C_3 \cos\gamma x \sin\gamma x) \\ &= (1-\lambda)\gamma^2(C_2^2 + C_3^2) + (\gamma^2-\gamma^4)C_2\cos\gamma x + (\gamma^2-\gamma^4)C_3 \sin\gamma x \in W_3. \end{aligned}$$

Now, we search an exact solution admitted by the time fractional (mKS) equation (4.1) of the form:

$$u(t,x) = C_1 + C_2 \cos\gamma x + C_3 \sin\gamma x. \tag{4.4}$$

Consequently, a function $u(t,x)$ of the form (4.4) is a solution of the time fractional (mKS) equation if the expansion coefficients $C_i(t)$ satisfy the following system of ordinary fractional differential equations:

$$\begin{cases} D_t^\alpha C_1 &= (1-\lambda)\gamma^2 C_3^2 + (1-\lambda)\gamma^2 C_2^2, \\ D_t^\alpha C_2 &= \gamma^2(1-\gamma^2)C_2, \\ D_t^\alpha C_3 &= \gamma^2(1-\gamma^2)C_3. \end{cases} \tag{4.5}$$

To get a non trivial solution needs to assume the condition $C_2(0)C_3(0) \neq 0$ and for convenience we suppose $C_2(0) = C_3(0) = 1$. This last condition will be clear when the Laplace transform will be used. We start to construct solution of the third equation in the above reduced system of ordinary fractional differential equations. We mention that, with the Laplace transform it is frequently possible to avoid working with equations of different differential orders by translating the problem into an easy one.

Recalling some useful properties of the Laplace transform [1]:

$$\mathfrak{L}\{D_t^\alpha f(t)\} = s^\alpha \widetilde{f}(s) - s^{\alpha-1} f(0), \quad 0<\alpha<1, \tag{4.6}$$

where

$$\mathfrak{L}\{f(t)\} = \widetilde{f}(s) = \int_0^\infty e^{-st} f(t)dt. \tag{4.7}$$

By putting $\theta = \gamma^2(1-\gamma^2)$ and applying the Laplace transform on both sides of the third equation appearing in the fractional ordinary differential system, we obtain:

$$s^\alpha \mathfrak{L}\{C_3(t)\} - s^{\alpha-1}C_3(0) = \theta \mathfrak{L}\{C_3(t)\}, \tag{4.8}$$

it yields:

$$\mathfrak{L}\{C_3(t)\}(s) = \frac{s^{\alpha-1}}{s^\alpha - \theta}, \tag{4.9}$$

then, with the inverse Laplace transform, it gives:

$$C_3(t) = E_{\alpha,1}(\theta t^\alpha), \tag{4.10}$$

where $E_{\alpha,\beta}(\theta t^\alpha)$ is the Mittag-Leffler function given by:

$$E_{\alpha,\beta}(\theta t^\alpha) = \Sigma_{i=0}^\infty \frac{\theta^i t^{\alpha i}}{\Gamma(\alpha i + \beta)}. \tag{4.11}$$

Not that when $\beta = 1$, $E_{\alpha,1} \equiv E_\alpha$.

Two last equations in the fractional ordinary differential system (4.5) are the same, hence,

$$C_2(t) = C_3(t) = E_{\alpha,1}(\theta t^\alpha). \tag{4.12}$$

Substituting the obtained expressions of C_2 and C_3 in the first equation of the system (4.5), it leads to:

$$D_t^\alpha C_1 = 2(1-\lambda)\gamma^2 \left(E_{\alpha,1}(\theta t^\alpha)\right)^2. \tag{4.13}$$

By using some useful properties of the Mittag-Leffler functions [16],[17],[18] the above relation, becomes:

$$D_t^\alpha C_1 = 2(1-\lambda)\gamma^2 E_{\alpha,1}(\theta(2t)^\alpha). \tag{4.14}$$

Applying J^α on both sides of equation (4.14), and using integration of the Mittag-Leffler function relation [1] (p. 25), we obtain:

$$\frac{1}{\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1} E_{\alpha,\beta}(\eta t^\alpha) t^{\beta-1} dt = t^{\beta+\nu-1} E_{\alpha,\beta+\nu}(\eta t^\alpha), \tag{4.15}$$

where $\alpha > 0$, $\beta > 0$ and $\nu > 0$, it leads by taking $\alpha = \nu,\ \eta = \theta 2^\alpha$ and $\beta = 1$ to:

$$J^\alpha E_{\alpha,1}(\theta(2t)^\alpha) = t^\alpha E_{\alpha,\alpha+1}(\theta(2t)^\alpha). \tag{4.16}$$

According to the following relation, it yields:

$$J^\alpha D_t^\alpha C_1(t) = C_1(t) - C_1(0), \qquad 0 < \alpha < 1, \tag{4.17}$$

we obtain

$$C_1(t) = 2(1-\lambda)\gamma^2 t^\alpha E_{\alpha,\alpha+1}(\theta(2t)^\alpha) + C_1(0). \tag{4.18}$$

We assume $C_1(0) = 0$. Hence, the obtained solution of fractional ordinary differential system (4.5) yields the following exact solution of the nonlinear time fractional modified Kuramoto-Sivashinsky equation (4.1):

$$u(t,x) = 2(1-\lambda)\gamma^2 t^\alpha E_{\alpha,\alpha+1}(\theta(2t)^\alpha) + E_{\alpha,1}(\theta t^\alpha)(\cos\gamma x + \sin\gamma x), \tag{4.19}$$

where $\gamma = \sqrt{\frac{1-\lambda}{\lambda}}$ and $\theta = \gamma^2 - \gamma^4$.

5 Some Particular Cases

In this section, we extract some particular cases, precisely exact solutions corresponding to $\lambda = \frac{1}{m}$, with $m \in \mathbb{N}^\star - \{1\}$.

Case 1 $\lambda = \frac{1}{2}$

This particular value of λ leads to $\gamma = 1$ and $\theta = 0$. Consequently, an exact solution of the studied fractional equation (4.1) is given by:

$$u(t,x) = \frac{t^\alpha}{\Gamma(\alpha+1)} + \cos x + \sin x,\ 0 < \alpha \leq 1. \tag{5.1}$$

Case 2 $\lambda = \frac{1}{m}$, $m > 2$

In this case we obtain that $\gamma = \sqrt{m-1}$ and $\theta = (m-1)(2-m)$. The constructed exact solution takes the form:

$$\begin{aligned} u_{\alpha,m}(t,x) &= 2^{1-\alpha}\frac{(m-1)}{m(2-m)}\left\{E_{\alpha,1}\left[(m-1)(2-m)2^\alpha t^\alpha\right] - 1\right\} \\ &+ E_{\alpha,1}\left[(m-1)(2-m)t^\alpha\right]\cos\sqrt{m-1}x \\ &+ E_{\alpha,1}\left[(m-1)(2-m)t^\alpha\right]\sin\sqrt{m-1}x. \end{aligned} \tag{5.2}$$

Now we look for solutions of nonlinear time fractional equation (4.1) corresponding to $\alpha = 1$ and $\alpha = \frac{1}{2}$.

Subcase 2.1 $\lambda = \frac{1}{m}$, $m > 2, \alpha = 1$.

According to the relation

$$E_{1,1}(z) = E_1(z) = e^z, \tag{5.3}$$

the corresponding exact solution of equation (4.1) is obtained to be of the following form:

$$\begin{aligned} u_{1,m}(t,x) &= \frac{m-1}{m(2-m)}\left\{e^{2(m-1)(2-m)t} - 1\right\} + e^{(m-1)(2-m)t}\cos\sqrt{m-1}x \\ &+ e^{(m-1)(2-m)t)}\sin\sqrt{m-1}x. \end{aligned} \tag{5.4}$$

Subcase 2.2 $\lambda = \frac{1}{m}$, $m > 2$, $\alpha = \frac{1}{2}$.

According to the relation:

$$E_{\frac{1}{2},1}(z) = E_{\frac{1}{2}}(z) = e^{z^2}\left(1 + erf(z)\frac{2}{\sqrt{\pi}}\int_0^z e^{-y^2}\ dy\right), \tag{5.5}$$

where

$$erf(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-y^2}\ dy, \tag{5.6}$$

the corresponding exact solution of equation (4.1) is obtained in this subcase to be of the form:

$$\begin{aligned} u_{\frac{1}{2},m}(t,x) &= \frac{\sqrt{2}(m-1)}{m(2-m)}\left\{E_{\frac{1}{2}}\left[\sqrt{2}(m-1)(2-m)\sqrt{t}\right] - 1\right\} \\ &+ e^{(m-1)(2-m)t}\left(\cos\sqrt{m-1}x + \sin\sqrt{m-1}x.\right). \end{aligned} \tag{5.7}$$

6 The Numerical Simulation

In this numerical simulation, six exact solutions of Eq(4.1) have been used to draw the graphs as shown in Figs 1-6 for different values of fractional parameter α and m.

Fig.1 : $\alpha = 1$ and $m = 3$	Fig.2 : $\alpha = 1$ and $m = 4$
Fig.3 : $\alpha = \frac{1}{2}$ and $m = 3$	Fig.4 : $\alpha = \frac{1}{2}$ and $m = 4$
Fig.5 : $\alpha = 1$ and $\lambda = \frac{1}{2}$	Fig.6 : $\alpha = \frac{1}{2}$ and $\lambda = \frac{1}{2}$

Here, we have drawn the corresponding 3-D surfaces for the obtained exact solutions of (mKS) equation in case of particular values of order derivatives.

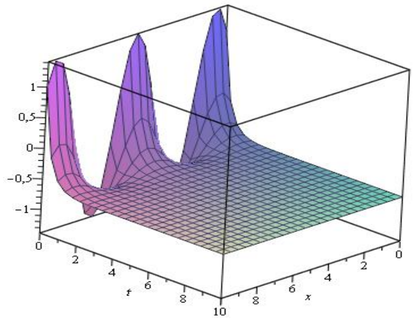

Fig. 1. Sol. $\alpha = 1$ **and** $m = 3$

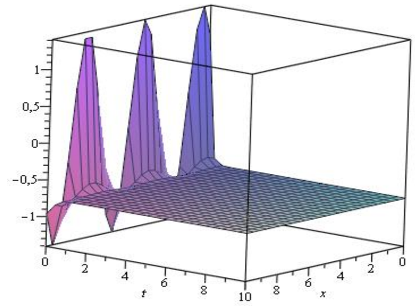

Fig. 2. Sol. $\alpha = 1$ **and** $m = 4$

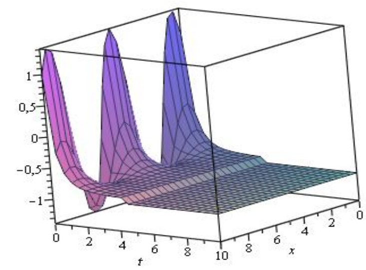

Fig. 3. Sol. $\alpha = \frac{1}{2}$ **and** $m = 3$

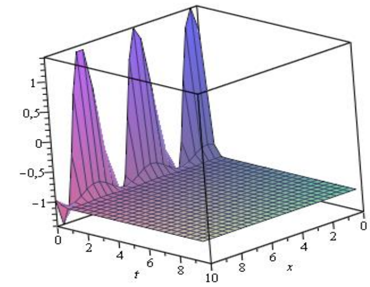

Fig. 4. Sol. $\alpha = \frac{1}{2}$ **and** $m = 4$

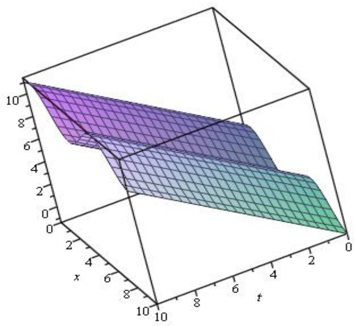

Fig. 5. Sol. $\alpha = 1$ **and** $\lambda = \frac{1}{2}$

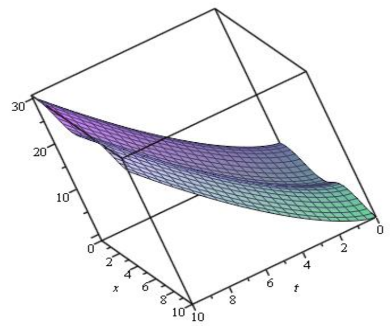

Fig. 6. Sol. $\alpha = \frac{1}{2}$ **and** $\lambda = \frac{1}{2}$

7 Conclusion

Here, by using the invariant subspace method to the time fractional nonlinear modified Kuramoto-Sivashinsky equation we obtain a nonlinear reduced system of fractional equations. The later was solved by the Laplace transform method and using some basic properties of the Mittag-Leffler functions. Some particular exact solutions of studied time fractional equation are given and their 3-D surfaces are drawn. Finally, we note that the construction of particular exact solutions of fractional differential equations is not an easy task until now and it remains a relevant problem. So, the method used in this paper for (mKS) equation can be extended to other nonlinear fractional differential equations.

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Competing Interests

Authors have declared that no competing interests exist.

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