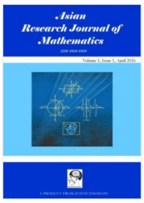

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A New Reconstraction Approach of Riccati Differential Equation for Solving a Class of Fractional Optimal Control Problems

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Authors' contributions

This work was carried out in collaboration between both authors. Author SSZ designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript and managed literature searches. Author MY managed the analyses of the study and literature searches. Both authors read and approved the final manuscript.

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Abstract

This paper presents a new approach for solving a class of linear quadratic fractional optimal control problems (FOCPs). The necessary optimality conditions for this problem are achieved in terms of two-point boundary value problem(TPBVP). In this way, an approximate approach is constructed based on solving a fractional Riccati differential equation (FRDE) such that the exact boundary conditions are satisfied. By solving this equation, we obtain the approximate solutions of the original problem.

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1 Introduction

Fractional calculus and fractional differential equations (FDEs), that are generalized the Calculus of variations [1],[2], use in basic sciences and engineering [3],[4],[5]. In this way, considerable effort has been made in developing solutions of FDEs [6],[7],[8],[9],[10]. Recently, the applications of this eqauations have included in various classes of FOCPs that refers to the minimization of a performance index subject to the FDEs are used as the dynamic constraints [11],[12],[13]. With the emerging number of the applications of FOCPs, the solution of these kind of problems has become an important topic for researchers. Using necessary optimality conditions, the FOCP reduced to a system of FDEs so that by finding its solution, one approximates the solutions of the original problem. A general formolation of FOCPs was extended by [14],[15], where the necessary conditions of optimization are achieved with the Caputo and Reimann-Liouvile derivatives. Since, it is difficult to obtain the exact solutions of FOCPs, approximate and numerical methods are used extensively that can be seen in [16],[17],[18], [19],[20].

In the present work, we developed an efficient and accurate approach for solving a class of FOCPs. The method we used here, consists of reducing the given FOCP to a system of FDEs such as TPBVP. Then, we approximated the fractional derivative operator by using the new formula that proposed in [21]. We apply this approach to develop some iterative formulas for solving TPBVP.

This study is organized as follows: In Section 2, some important definitions and necessary preliminaries of fractional derivatives are described. We summarize the necessary optimality conditions of FOCPs and the reconstraction approach of it's solutions in Section 3. In Section4, the approximation is applied to some examples to show the efficiency and simplicity of our approach.

2 Basic Definitions and Properties of Fractional Derivative

In this section, we briefly give some definitions regarding fractional derivatives allowing us to formulate a general definition of an FOCP. The most important types of fractional derivatives are Riemann-Liouville (RLFD) and Caputo fractional derivatives (CFD) that we adopt here the Caputo definition. For the definitions of fractional derivatives and some of their applications, see [22],[23],[24],[25].

Definition 2.1. $\Gamma : (0, \infty) \to \mathbb{R}$ is known as the Euler-Gamma function (or Euler integral of the second kind) and defined by:

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^t dt. \tag{2.1}$$

Definition 2.2. The left fractional integral operator of order $\alpha > 0$ of a function $f \in L_1([t_0, t_f], \mathbb{R}^n)$, is defined as:

$$_{t_0}I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \tag{2.2}$$

and the right fractional integral has the following definition:

$$_tI_{t_f}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^{t_f} (t-\tau)^{\alpha-1} f(\tau) d\tau. \tag{2.3}$$

It is identity that $_{t_0}I_t^0 f(t) = {}_tI_{t_f}^0 f(t) = f(t)$.

Definition 2.3. The left CFD and the right CFD of order $\alpha \in \mathbb{R}$ of a continuously differentiable function $f(t)$ are given by:

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} \Big(\frac{d}{d\tau}\Big)^n f(\tau) d\tau, \tag{2.4} $$

$$ {}_{t}^{C}D_{t_f}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t}^{t_f} (\tau-t)^{n-\alpha-1} \Big(\frac{-d}{d\tau}\Big)^n f(\tau) d\tau, \tag{2.5} $$

where $n-1 < \alpha \leq n$ and $n \in \mathbb{N}$. When α is an integer, the usual definitions of the derivatives are considered.

Some useful properties of fractional integrals and derivatives, include all fractional operators are linear, that is, if D is an arbitrary fractional operator and c, d are two arbiterary constants, then:

$$ D(cf + dg) = cD(f) + dD(g). \tag{2.6} $$

Also, for all function $f \in C^n[t_0, t_f]$ and $n \in \mathbb{N}$; if $\alpha, \beta > 0$, then:

$$ I^{\alpha} I^{\beta} f = I^{\alpha+\beta} f, \;\; D^{\alpha} D^{\beta} f = D^{\alpha+\beta} f. \tag{2.7} $$

The following theorem, helps us to apply a fractional integral over a fractional derivative:

Theorem 2.1. *Let $\alpha, \beta > 0$ and $n = \lceil \alpha \rceil$. If $f(t) \in C^n[t_0, t_f]$; Then*

$$ \begin{aligned} {}_{t_0}^{C}D_t^{-\alpha} \Big({}_{t_0}^{C}D_t^{\beta} f(t) \Big) &= {}_{t_0}^{C}D_t^{-\alpha+\beta} f(t), \\ {}_{t_0}^{C}D_t^{-\alpha} \Big({}_{t_0}^{C}D_t^{\alpha} f(t) \Big) &= f(t) - \sum_{k=1}^{n} {}_{t_0}^{C}D_t^{\alpha-k} f(t) \Big|_{t=t_0} \frac{(t-t_0)^{\alpha-k}}{\Gamma(\alpha-k+1)}. \end{aligned} $$

In particular, if $0 < \alpha \leq 1$ and $f(t) \in C[t_0, t_f]$, then:

$$ {}_{t_0}^{C}D_t^{-\alpha} \Big({}_{t_0}^{C}D_t^{\alpha} f(t) \Big) = f(t) - f(t_0). $$

Later, for computational purposes, a new expanision formula was obtained in [21], which is equivalent to the fractional derivatives:

$$ \begin{aligned} {}_{t_0}^{C}D_t^{\alpha} x(t) \simeq\; & A(\alpha, N)(t-t_0)^{-\alpha} x(t) + B(\alpha, N)(t-t_0)^{1-\alpha} \dot{x}(t) \\ & - \sum_{p=2}^{N} C(\alpha, N)(t-t_0)^{1-p-\alpha} V_p(t) - \frac{x(t_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}, \end{aligned} \tag{2.8} $$

where $V_p(t)$ is defined as the solution of the system

$$ \begin{cases} \dot{V}_p(t) = (1-p)(t-t_0)^{p-2} x(t), \\ V_p(t_0) = 0, \quad p = 2, 3, \cdots, N, \end{cases} $$

and

$$ \begin{aligned} {}_{t}^{C}D_{t_f}^{\alpha} x(t) \simeq\; & A(\alpha, N)(t_f-t)^{-\alpha} x(t) - B(\alpha, N)(t_f-t)^{1-\alpha} \dot{x}(t) \\ & + \sum_{p=2}^{N} C(\alpha, N)(t_f-t)^{1-p-\alpha} W_p(t) - \frac{x(t_f)(t_f-t)^{-\alpha}}{\Gamma(1-\alpha)}, \end{aligned} \tag{2.9} $$

where $W_p(t)$ is the solution of the differential equation

$$\begin{cases} \dot{W}_p(t) = -(1-p)(t_f - t)^{p-2}x(t), \\ W_p(t_f) = 0, \quad p = 2, 3, \cdots, N, \end{cases}$$

and $A(\alpha, N)$, $B(\alpha, N)$, $C(\alpha, p)$ are defined by:

$$A(\alpha, N) = \frac{1}{\Gamma(1-\alpha)}\Big[1 + \sum_{p=2}^{N} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha)(p-1)!}\Big],$$

$$B(\alpha, N) = \frac{1}{\Gamma(2-\alpha)}\Big[1 + \sum_{p=2}^{N} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}\Big],$$

$$C(\alpha, p) = \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha-1)} \frac{\Gamma(p-1+\alpha)}{(p-1)!}, \quad p = 2, 3, \cdots, N.$$

We use the Caputo fractional derivative because it computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative and also, it allows traditional (integer order) initial and boundary conditions to be included in the formulation of the problem.

3 Formulation of The Fractional Optimal Control Problem

More specifically, an important contribution of the work presented in this paper, is the fact that we present a formulation and a numerical scheme for solving FOCP based on FRDE. For this purpose, let $\alpha \in (0, 1)$, $t_0, t_f \in \mathbb{R}$. Consider the linear fractional system:

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} x(t) = a(t)x(t) + b(t)u(t) \\ \qquad x(t_0) = x_0, \end{cases} \tag{3.1}$$

with the following cost functional:

$$\min J(u(.)) = \frac{1}{2}\int_{t_0}^{t_f} \Big\{x^T(t)q(t)x(t) + u^T(t)r(t)u(t)\Big\}dt, \tag{3.2}$$

where it is assumed that J is of class C^1, $x(t)$, $u(t)$ are the n-dimensional state vector and the m-dimensional control vector, respectively, $a(t)$, $b(t)$ are matrices of appropratie dimentions, $q(t)$ is a symmetric and positive-semidefinite matrices, $r(t)$ is a symmetric and positive-definite matrix and x_0 is a fixed real number.

The aim is to find a control vector $u^*(t)$ such that the cost functional (3.2) is minimized while the dynamic equality constraint (3.1) is satisfied. To obtain the necessary conditions, we define the Hamiltonian:

$$H(\lambda, t) = \frac{1}{2}\Big(x^T(t)q(t)x(t) + u^T(t)r(t)u(t)\Big) + \lambda^T\Big(a(t)x(t) + b(t)u(t)\Big), \tag{3.3}$$

where λ is the vector of the Lagrange multiplier. By application of the maximum principle for problem (3.1)-(3.2) we can obtain the following nonlinear TPBVP [26]:

$$
\begin{aligned}
{}_{t}^{C}D_{t_f}^{\alpha}\lambda(t) &= \frac{\partial H}{\partial x} = q(t)x(t) + a^T(t)\lambda(t),\ \lambda(t_f) = 0 \\
{}_{t_0}^{C}D_{t}^{\alpha}x(t) &= \frac{\partial H}{\partial \lambda} = a(t)x(t) + b(t)u(t),\ x(t_0) = x_0 \\
0 &= \frac{\partial H}{\partial u} = r(t)u(t) + b^T(t)\lambda(t).
\end{aligned} \tag{3.4}
$$

From this system of equations we obtain $u(t) = -r^{-1}(t)b^T(t)\lambda(t)$. So, it can be demonstrated that the necessary conditons for system (3.1)-(3.2) are as follows:

$$
\begin{aligned}
{}_{t}^{C}D_{t_f}^{\alpha}\lambda(t) &= q(t)x(t) + a^T(t)\lambda(t) \\
{}_{t_0}^{C}D_{t}^{\alpha}x(t) &= a(t)x(t) - b(t)r^{-1}(t)b^T(t)\lambda(t) \\
\lambda(t_f) &= 0,\ \ x(t_0) = x_0.
\end{aligned} \tag{3.5}
$$

3.1 Reconstraction of the optimality conditions

TPBVP (3.5) will be used to develop the numerical solutions of FOCP (3.1)-(3.2). In general, no analytical solution of equations (3.5) exists. Therefore, we apply a new reconstraction approach to solve this problem. At first, by substituting relations (2.8) and (2.9) in equations (3.5) we will have:

$$\dot{X}(t) = A(t)X(t) - B(t)R^{-1}(t)B^T(t)\Lambda(t) + F(t) \tag{3.6}$$

$$-\dot{\Lambda}(t) = Q(t)X(t) + D(t)\Lambda(t), \tag{3.7}$$

in which,

$$
A(t) = \begin{pmatrix}
\dfrac{a(t) - A(\alpha,N)(t-t_0)^{-\alpha}}{B(\alpha,N)(t-t_0)^{1-\alpha}} & \dfrac{C(\alpha,2)(t-t_0)^{1-2-\alpha}}{B(\alpha,N)(t-t_0)^{1-\alpha}} & \dfrac{C(\alpha,N)(t-t_0)^{1-N-\alpha}}{B(\alpha,N)(t-t_0)^{1-\alpha}} \\
(1-2)(t-t_0)^{2-2} & 0 \quad \cdots & 0 \\
\vdots & \vdots & \vdots \\
(1-N)(t-t_0)^{N-2} & 0 \quad \cdots & 0
\end{pmatrix}
$$

$$
D(t) = \begin{pmatrix}
\dfrac{a^T(t) - A(\alpha,N)(t_f-t)^{-\alpha}}{B(\alpha,N)(t_f-t)^{1-\alpha}} & \dfrac{-C(\alpha,2)(t_f-t)^{1-2-\alpha}}{B(\alpha,N)(t_f-t)^{1-\alpha}} & \dfrac{-C(\alpha,N)(t_f-t)^{1-N-\alpha}}{B(\alpha,N)(t_f-t)^{1-\alpha}} \\
(1-2)(t_f-t)^{2-2} & 0 \quad \cdots & 0 \\
\vdots & \vdots & \vdots \\
(1-N)(t_f-t)^{N-2} & 0 \quad \cdots & 0
\end{pmatrix}
$$

$$B(t) = \begin{pmatrix} \dfrac{b(t)}{B(\alpha, N)(t-t_0)^{1-\alpha}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad F(t) = \begin{pmatrix} \dfrac{x(t_0)}{\Gamma(1-\alpha)B(\alpha, N)(t-t_0)} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$Q(t) = \begin{pmatrix} q(t) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad R(t) = r(t),$$

$$X(t) = \begin{pmatrix} x(t) \\ V_2(t) \\ \vdots \\ V_N(t) \end{pmatrix}, \quad X(t_0) = \begin{pmatrix} x_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \Lambda(t) = \begin{pmatrix} \lambda(t) \\ W_2(t) \\ \vdots \\ W_N(t) \end{pmatrix}, \quad \Lambda(t_f) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.8}$$

In order to solve the nonlinear TPBVP (3.5), it is sufficient to solve equations (3.6)-(3.7). Let:

$$\Lambda(t) = P(t)X(t) + G(t), \quad \Lambda(t_f) = 0, \tag{3.9}$$

where $P(t)$ is a symmetric matrix. By calculating the derivatives to the both sides of equation (3.9) and using the equations (3.7) we have:

$$\begin{aligned} \dot{\Lambda}(t) &= \dot{P}(t)X(t) + P(t)\dot{X}(t) + \dot{G}(t) \\ &= \dot{P}(t)X(t) + P(t)A(t)X(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t)X(t) \\ &\quad - P(t)B(t)R^{-1}(t)B^T(t)G(t) + P(t)F(t) + \dot{G}(t). \end{aligned} \tag{3.10}$$

Now, with comparison of equations (3.7) and (3.10), we will have:

$$\begin{aligned} 0 = &\Big\{\dot{P}(t) + P(t)A(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t) + D(t)P(t) + Q(t)\Big\}X(t) \\ &+ \Big(\dot{G}(t) + D(t)G(t) - P(t)B(t)R^{-1}(t)B^T(t)G(t) + P(t)F(t)\Big). \end{aligned} \tag{3.11}$$

Now, the relation (3.11) must satisfy for all $X(t)$ and t, which leads us to the $N \times N$ matrix $P(t)$ in which satisfy the Riccati matrix differential equation:

$$-\dot{P}(t) = P(t)A(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t) + D(t)P(t) + Q(t), \quad P(t_f) = 0, \tag{3.12}$$

and the following adjoint vector differential equation:

$$\dot{G}(t) = -D(t)G(t) + P(t)B(t)R^{-1}(t)B^T(t)G(t) - P(t)F(t), \tag{3.13}$$

with boundary condition $G(t_f) = 0$. Also, by Substituting (3.9) into equation (3.6), we can get the following optimal closed loop system:

$$\dot{X}(t) = \Big(A(t) - B(t)R^{-1}(t)B^T(t)P(t)\Big)X(t) - B(t)R^{-1}(t)B^T(t)G(t) + F(t), \tag{3.14}$$

such that the boundary condition $X(t_0)$ given by (3.8).

Theorem 3.1. *For the nonlinear system described by* (3.1)-(3.2)*, the optimal control law at time t is uniquely given by:*

$$u^*(t) = -R^{-1}(t)B^T(t)\Big(P(t)X(t) + G(t)\Big), \tag{3.15}$$

where, $P(t)$ and $G(t)$ are the solutions of differential equations (3.12) *and* (3.13)*, respectively. Moreover, minimum value of the cost functional $J(u(.))$ is given by:*

$$J^*(u(.)) = \frac{1}{2}X^T(t)P(t)X(t) + X^T(t)G(t) + H(t), \tag{3.16}$$

where $X(t)$ is the solutions of differential equation (3.14) *and $H(t)$ is determined from:*

$$\dot{H}(t) = \frac{1}{2}G^T BR^{-1}B^T G - F^T G - \frac{1}{2}X^T(A^T - D)(PX + G), \tag{3.17}$$

with the final condition $H(t_f) = 0$.

Proof. By direct substitution of $u(t)$ into the performance index (3.2), with the initial time replaced by t and the final time by t_f with substitution $Q(t)$ from (3.12) we obtain:

$$\begin{aligned} J(u(.)) = \frac{1}{2}\int_t^{t_f} \Big(&X^T(\tau)\Big\{ - \dot{P}(\tau) - P(\tau)A(\tau) - D(\tau)P(\tau) - 2p(\tau)S(\tau)P(\tau)\Big\}X(\tau) \\ &+ X^T(\tau)P(\tau)S(\tau)G(\tau) + G^T(\tau)S(\tau)P(\tau)X(\tau) + G^T(\tau)S(\tau)G(\tau)\Big)d\tau, \end{aligned} \tag{3.18}$$

where in, $S(t) = B(t)R^{-1}(t)B^T(t)$. Now, with substitution $X^T(D - PS)$ from (3.13), we have:

$$\begin{aligned} J(u(.)) &= \frac{1}{2}\int_t^{t_f} \Big(- X^T(\tau)\dot{P}(\tau)X(\tau) - X^T(\tau)P(\tau)\dot{X}^T(\tau) - \dot{X}^T(\tau)P(\tau)X(\tau) \\ &- 2X^T(\tau)\dot{G}(\tau) - 2\dot{X}^T(\tau)G(\tau) + 2F^T(\tau)G(\tau) - G^T(\tau)S(\tau)G(\tau) \\ &+ X^T(\tau)\Big(A^T(\tau) - D(\tau)\Big)P(\tau)X(\tau) + X^T(\tau)\Big(A^T(\tau) - D(\tau)\Big)G(\tau)\Big)d\tau \\ &= \Big[- \frac{1}{2}X^T(\tau)P(\tau)X(\tau) - X^T(\tau)G(\tau)\Big]\Big|_t^{t_f} + \frac{1}{2}\int_t^{t_f} \Big(- G^T(\tau)S(\tau)G(\tau) \\ &+ 2F^T(\tau)G(\tau) + X^T(\tau)\Big(A^T(\tau) - D(\tau)\Big)\Big(P(\tau)X(\tau) + G(\tau)\Big)\Big)d\tau. \end{aligned} \tag{3.19}$$

Therefore the results are clearly obtained. □

3.2 Numerical approach

It is well known that the analytical solution of the TPBVP (3.5) does not exist generally. Therefore, it is necessary to find approximation approaches for solving this problem. So, we introduce a sequence of TPBVPs as [27]:

$$\dot{X}^{(k)}(t) = A(t)X^{(k)}(t) - B(t)R^{-1}(t)B^T(t)\Lambda^{(k)}(t) + F(t) \tag{3.20}$$

$$-\dot{\Lambda}^{(k)}(t) = Q(t)X^{(k)}(t) + D(t)\Lambda^{(k)}(t), \tag{3.21}$$

The control sequance is given by $u^{*(k)}(t) = -R^{-1}(t)B^T(t)\Lambda^{(k)}(t)$ in which:

$$\Lambda^{(k)}(t) = P(t)X^{(k)}(t) + G^{(k)}(t), \;\; \Lambda(t_f) = 0. \tag{3.22}$$

Also we obtain the Riccati matrix differential equation by (3.12) and the adjoint vector differential equation from:

$$\dot{G}^{(k)}(t) = -D(t)G^{(k)}(t) + P(t)B(t)R^{-1}(t)B^T(t)G^{(k)}(t) - P(t)F(t), \;\; G^{(k)}(t_f) = 0, \tag{3.23}$$

and then we will have:

$$\dot{X}^{(k)}(t) = \Big(A(t) - B(t)R^{-1}(t)B^T(t)P(t)\Big)X^{(k)}(t) - B(t)R^{-1}(t)B^T(t)G^{(k)}(t) + F(t), \tag{3.24}$$

with the boundary condition $X^{(k)}(t_0) = X^{(k-1)}(t_0)$.

Assuming $X^{(0)}(t) = 0$ and $\Lambda^{(0)}(t) = 0$, are considered as initial approximation. Then, $G^{(k)}(t)$ and $X^{(k)}(t)$, $k \geq 1$, can be obtained from solving the vector differential equations (3.23) and (3.24), simultaneously, that are uniformly converge to the solution of vector differential equations (3.13) and (3.14), [28], respectively. So, It is clear that the control sequence $u^{(k)}(t)$ is also uniformly convergent. Moreover, The minimum value of $J^{(k)}(u(.))$ is given by:

$$J^{(k)}(u(.)) = \frac{1}{2}(X^{(k)})^T(t)P(t)X^{(k)}(t) + (X^{(k)})^T(t)G^{(k)}(t) + H^{(k)}(t), \tag{3.25}$$

where, $H^{(k)}(t)$ is the sequence solution of:

$$\dot{H}^{(k)}(t) = \frac{1}{2}(G^{(k)})^T BR^{-1}BG^{(k)} - F^T G^{(k)} - \frac{1}{2}(X^{(k)})^T(A^T - D)\Lambda^{(k)}, \tag{3.26}$$

with the final condition $H^{(k)}(t_f) = 0$.

4 Numerical Examples

In this section, to demonstrate the applicability of the formulation, we present numerical results of following FOCPs.

Example 4.1. *Consider the following FOCP:*

$$\min J(u(.)) = \frac{1}{2}\int_0^1 \Big(x^2(t) + u^2(t)\Big)dt \tag{4.1}$$

subject to

$$ {}_0^C D_t^\alpha x(t) = -x(t) + u(t), \;\; x(0) = 1. \tag{4.2}$$

The necessary conditions for this problem are as follows [29]:

$$\begin{aligned} &{}_t^C D_1^\alpha \lambda(t) = x(t) + u(t) \\ &{}_0^C D_t^\alpha x(t) = -x(t) - \lambda(t) \\ &u(t) + \lambda(t) = 0 \\ &\lambda(1) = 0, \;\; x(0) = 1. \end{aligned} \tag{4.3}$$

This is a common time independent problem when $\alpha = 1$. Optimal state and optimal control of above problem when $\alpha = 1$ is as following:

$$x(t) = cosh(\sqrt{2}t) + \beta sinh(\sqrt{2}t), \tag{4.4}$$

and

$$u(t) = (1 + \beta\sqrt{2})cosh(\sqrt{2}t) + \beta\sqrt{2}sinh(\sqrt{2}t), \tag{4.5}$$

where:

$$\beta = -\frac{cosh(\sqrt{2}) + \sqrt{2}sinh(\sqrt{2})}{\sqrt{2}cosh(\sqrt{2}) + sinh(\sqrt{2})} \cong -0.9799$$

Now, we can use our approximation to solve this problem. Note that, for this example, we have $q(t) = r(t) = -a(t) = b(t) = x_0 = 1$, $\alpha = 0.5$. By substituting relations (2.8) and (2.9) into equations (4.3) and assuming $N = 2$, $X^T(t) = [x(t), V_2(t)]$, $\Lambda^T(t) = [\lambda(t), W_2(t)]$, the result will be equations (3.6) and (3.7) where in:

$$A(t) = \begin{pmatrix} \dfrac{-1 - 0.8463 \times t^{-0.5}}{0.4231 \times t^{0.5}} & \dfrac{0.2821}{0.4231 \times t^{2}} \\ -1 & 0 \end{pmatrix}, \; B(t) = \begin{pmatrix} \dfrac{1}{0.4231 \times t^{0.5}} \\ 0 \end{pmatrix},$$

$$D(t) = \begin{pmatrix} \dfrac{-1 - 0.8463 \times (1-t)^{-0.5}}{0.4231 \times (1-t)^{0.5}} & \dfrac{0.2821}{0.4231 \times (1-t)^{2}} \\ -1 & 0 \end{pmatrix}$$

$$F(t) = \begin{pmatrix} \dfrac{1}{0.1221 \times t} \\ 0 \end{pmatrix}, \; Q(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \Lambda(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then, we will have:

$$S(t) = \begin{pmatrix} \dfrac{1}{0.1790 \times t} & 0 \\ 0 & 0 \end{pmatrix}.$$

Following the computational steps outlined above, the Riccati matrix differential equation (3.12) and the adjoint vector differential equation (3.13) becomes as follows:

$$\begin{aligned} -\dot{P}(t) = {} & P(t) \begin{pmatrix} \dfrac{-1 - 0.8463 \times t^{-0.5}}{0.4231 \times t^{0.5}} & \dfrac{0.2821}{0.4231 \times t^{2}} \\ -1 & 0 \end{pmatrix} - P(t) \begin{pmatrix} \dfrac{1}{0.1790 \times t} & 0 \\ 0 & 0 \end{pmatrix} P(t) \\ & + \begin{pmatrix} \dfrac{-1 - 0.8463 \times (1-t)^{-0.5}}{0.4231 \times (1-t)^{0.5}} & \dfrac{0.2821}{0.4231 \times (1-t)^{2}} \\ -1 & 0 \end{pmatrix} P(t) + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{4.6}$$

$$\begin{aligned} \dot{G}(t) = {} & \begin{pmatrix} \dfrac{1 + 0.8463 \times (1-t)^{-0.5}}{0.4231 \times (1-t)^{0.5}} & \dfrac{-0.2821}{0.4231 \times (1-t)^{2}} \\ 1 & 0 \end{pmatrix} G(t) - P(t) \begin{pmatrix} \dfrac{1}{0.1221 \times t} \\ 0 \end{pmatrix} \\ & + P(t) \begin{pmatrix} \dfrac{1}{0.1790 \times t} & 0 \\ 0 & 0 \end{pmatrix} G(t) \end{aligned} \tag{4.7}$$

in which,

$$P(1)=\begin{pmatrix}0&0\\0&0\end{pmatrix},\ G(1)=\begin{pmatrix}0\\0\end{pmatrix}.$$

Also, we have:

$$\dot{X}(t)=\left\{\begin{pmatrix}\dfrac{-1-0.8463\times t^{-0.5}}{0.4231\times t^{0.5}} & \dfrac{0.2821}{0.4231\times t^{2}}\\ -1 & 0\end{pmatrix}-\begin{pmatrix}\dfrac{1}{0.1790\times t} & 0\\ 0 & 0\end{pmatrix}P(t)\right\}X(t)$$

$$-\begin{pmatrix}\dfrac{1}{0.1790\times t} & 0\\ 0 & 0\end{pmatrix}G(t)+\begin{pmatrix}\dfrac{1}{0.1221\times t}\\ 0\end{pmatrix},\ X(0)=\begin{pmatrix}1\\0\end{pmatrix}. \tag{4.8}$$

The absolute errors of the cost functional values at different iteration steps are listed in Table 1. This value compares well with those given in [30]. From this table, it is observed that, the convergence is achieved only after three iterations and a minimum value of $J^{(4)}=6.15125$ is obtained. Therefore, $u^{(4)}$ can be approximately considered as the optimal control law u^{*}. Absolute errors of the optimal control $u(t)$ and the corresponding state $x(t)$ are depicted in Table 2. It can be seen that when iterations are increased, the better approximations to both the state and the control functions and than the better approximation of the optimal cost will be obtained.

Table 1. Absolute errors of cost functional values at the different iteration times

Iteration time k	1	2	3	4
Cost functional J	0.1041	0.0232	0.0033	0.0019

Table 2. Absolute errors of the optimal control and optimal state at different values of k

k	$u(t)$	$x(t)$
1	4.21340×10^{-2}	3.40323×10^{-3}
2	2.51714×10^{-4}	2.21262×10^{-4}
3	1.32072×10^{-7}	2.64992×10^{-6}
4	3.97889×10^{-9}	1.43634×10^{-9}

5 Conclusions

In the present work, we developed a new method for solving a class of FOCPs, by using TPBVP and different form of FRDE. The approach is computationally attractive and also reduces keeping the accuracy of the solution.

Competing Interests

Authors have declared that no competing interests exist.

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