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Investor's Power Utility Optimization with Consumption, Tax, Dividend and Transaction Cost under Constant Elasticity of Variance Model

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

This work considered an investor's portfolio where consumption, taxes, transaction costs and dividends are in involved, under constant elasticity of variance (CEV). The stock price is assumed to be governed by a constant elasticity of variance CEV model and the goal is to maximize the expected utility of consumption and terminal wealth where the investor has a power utility preference. The application of dynamic programming principles, specifically the maximum principle obtained the Hamilton-Jacobi-Bellman (HJB) equation for the value function on which elimination of variable dependency was applied to obtain the close form solution of the optimal investment and consumption strategies. It is found that optimal investment on the risky asset is horizon dependent.

Keywords: Investor; power utility; consumption; transaction costs; constant elasticity of variance; optimization.

1 Introduction

An investor dynamically allocates his wealth between a risk asset and a risk-free asset and chooses an optimal consumption rate to maximize total expected discounted utility of consumption, in the classical Merton's portfolio optimization problems [1,2]. In the Merton's model, there are no transaction costs and

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borrowing constraints and no shorting constraints. The investment and consumption problems have inspired literally hundreds of extensions and applications, since the pioneer work of Merton. Introducing transaction costs into the investment and consumption problems, are the contributions of Shreve and Soner [3], Akian et al. [4], and Janeˇcek and Shreve [5], for example. Among authors who investigated the optimal consumption problem with borrowing constraints are; Fleming and Zariphopoulou [6], Vila and Zariphopoulou [7], and Yao and Zhang [8]. These mentioned models generally were studied under the assumption that risky asset price dynamics was governed by a geometric Brownian motion (GBM).

A natural extension of the GBM is the constant elasticity of variance (CEV) model. The advantages of the CEV model as compared to the geometric Brownian motion (GBM), are that the volatility rate has correlation with the risky asset price and can explain volatility smile (a common graph shape that results from plotting the strike price and implied volatility of a group of options with the same expiration date. The volatility smile is so named because it looks like a person smiling). Cox and Ross [9] originally proposed the CEV model as an alternative diffusion process for European option pricing. Among contributors who applied the CEV model to analyze the option pricing formula are; Schroder [10], Lo et al. [11], Phelim and Yisong [12], and Davydov and Linetsky [13].

Recently, Xiao et al. [14], Gao [15,16], Gu et al. [17], Lin and Li [18], Gu et al. [19], Jung and Kim [20] and Zhao and Rong [21], have introduced the CEV model into annuity contracts and the optimal investment strategies in the utility framework applying dynamic programming principle.

The HJB equation derived is more difficult to deal with due to the introduction of consumption factor and the CEV model as compared to that obtained by Gao [15]. Inspired by the techniques of Gao [15] and Liu [22], one can transform the nonlinear second-order partial differential equation into a linear one using elimination of dependency on the variables W and S (wealth and price of the risky asset).

In this paper, CEV model is introduced in an investment and consumption problem, involving transaction costs and dividend, and optimally allocate the wealth between a risk-free asset and a risky asset, whose price process is supposed to follow a CEV model. The goal is to maximize the expected discounted utility of consumption and terminal wealth when transaction costs and dividends are involved. Dynamic programming principle, specifically the maximum principle, is applied to obtain the HJB equation for the value function.

The rest of this paper is organized as follows. Section 2 gives the model formulation and the model for the proposed optimization problem. In Section 3, the necessary theorem is stated and applying the dynamic programming principle (maximum principle) obtained the HJB equation and investigated the optimal investment and consumption strategies in the case of power utility preference. Section 4, gives the findings and section 5concludes the paper.

2 Model Formulation and the Model

In the financial market being considered, an investor is assumed to trade two assets: a riskless asset (bond) and a risky asset. The evolution of the price of the riskless asset denoted by $B(t)$ is governed by

$$dB(t) = rdt; \quad B(0) = 1, \tag{1}$$

and that of the risky stock, $S(t)$, described by the constant elasticity of variance (CEV) model (Gao [15]),

$$dS(t) = S(t)\,[\mu dt + bS^{\gamma}(t)dZ(t)], \tag{2}$$

where $S(t)$ is the price of the risky asset and $r\ \mu$, b, and γ are constants. μ is the appreciation rate of the risky asset and $bS^{\gamma}(t)$ it's volatility. $\{Z(t); t > 0\}$ is a standard Brownian motion in a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t>0}, \wp)$, $\mathcal{F}_{t>0}$ is the augmented filtration generated by the Brownian motion $Z(t)$ and γ the elasticity parameter satisfying the general condition $\gamma \leq 0$. When $\gamma = 0$, equation (2) reduces to geometric Brownian motion.

Let V(t) be the total amount of money the investor has available for investment and $\pi(t)$ the amount he has decided to invest on the risky asset, then $[\mathrm{V(t)} - \pi(t)]$ is the amount left for investment in the riskless asset (bond).

In the financial market, it is assumed that taxes, dividends and transaction costs are charged at the constant rates, $\lambda, \emptyset, and\ \theta$ respectively.

Assumptions:

1. Dividends are paid on the risky asset only.
2. Transaction costs and taxes are also charged on the risky asset only.
3. Consumption withdrawals are made on the riskless bond only.

The dynamics of the investor's wealth process corresponding to his trading strategy $\pi(t)$ is given by the stochastic differential equation (SDE)

$$dV(t) = \pi(t)\frac{ds(t)}{s(t)} + [\mathrm{V(t)} - \pi(t)]\frac{dB(t)}{B(t)} + [\emptyset - (\lambda + \theta)]\pi(t)dt - C(t)dt \tag{3}$$

Substituting (1) and (2) into (3) obtains,

$$dV(t) = \pi(t)[\mu dt + bS^{\gamma}(t)dZ(t)] + [\mathrm{V(t)} - \pi(t)]\, rdt + [\emptyset - (\lambda + \theta)]\, \pi(t)dt - C(t)dt \tag{4}$$

which simplifies to;

$$dV(t) = \{[(\mu + \emptyset) - (r + \lambda + \theta)]\, \pi(t) + r\, \mathrm{V(t)} - C(t)\}dt + b\pi(t)S^{\gamma}(t)dZ(t) \tag{5}$$

The quadratic variation of the wealth process is given by;

$$< dV(t) > = b^2\pi^2(t)S^{2\gamma}(t)dt \tag{6a}$$

where

$$dt.dt = dt.dZ(t) = 0;\ \ dZ(t).dZ(t) = dt. \tag{6b}$$

Assuming the investment strategy $\pi(t)$ is admissible, that is, $\pi(t)$ is $\mathcal{F}_{t>0}$ measurable, and

$$\mathrm{E}\left(\int_0^t \pi^2(t)\, dt\right) < \infty\, , \tag{7}$$

and the stochastic differential equation (5) has a unique solution for all $\pi(t)$, then the optimal strategy that maximizes the investor's expected power utility of terminal wealth is sort considering the wealth process described by (5) under the set of admissible process $\pi(t)$.

That is

$$Max_{\pi(t)}E\big[U\big(V(T)\big)\big] \tag{8}$$

where U (.) is strictly concave and satisfies the condition $U^{'}(+\infty) = 0$ and $U^{'}(0) = +\infty$ and T is the time horizon. This work considers the power utility function given by

$$U\big(V(t)\big) = \frac{V^{1-k}}{1-k}, k < 1. \tag{9}$$

Based on the classical tools of the stochastic optimal control where consumption is involved, define the value function at time T as;

$$J(T,S,V) = sup_{\pi} E\left[\int_0^{-T} e^{-\beta\tau} \frac{C^{1-k}}{1-k} dt + e^{-\beta T} \frac{V_T^{1-k}}{1-k}\right]; \mathrm{S(t) = s, V(t) = v}\,;\ 0 < t < T \tag{10}$$

Therefore, the investor's problem becomes;

$$J(t,s,v) = sup_{\pi} E\left[\int_0^{-T} e^{-\beta\tau} \frac{C^{1-k}}{1-k} dt + e^{-\beta T} \frac{V_T^{1-k}}{1-k}\right]$$

Subject to;

$$dv = \left[[(\mu + \emptyset) - (r + \lambda + \theta)]\,\pi + [r\,\mathrm{v} - C]dt + b\pi s^{\gamma} dZ(t)\right]$$

3 The Optimization Programme

The theorem below follows

Theorem: If an investor has a utility preference given by, $U(v) = \frac{V^{1-k}}{1-k}, k < 1,$ then the optimal policy that maximizes the expected utility at terminal time T is to invest in the risky asset;

$$\pi^* = \frac{v}{k}\left\{\frac{[(\mu+\emptyset)-(r+\lambda+\theta)]}{(bs^{\gamma})^2} + s\right\},$$

with optimal consumption;

$$C^* = \frac{v}{\frac{s^{1-k}}{1-k}\left\{e^{(n-1)\left[p(T-t)+\frac{q}{2}[T^2-t^2]\right]}\left[(1-n)\int_t^T e^{(1-n)\left[p(T-t)+\frac{q}{2}(T^2-t^2)\right]}dt + \left[\frac{1-k}{s^{1-k}}\right]^{1-n}\right]\right\}^{\frac{1}{1-n}}},$$

and optimal value function;

$$J^*(t,s,v) = \left\{\left[\frac{v^{1-k}}{1-k}\right]\left[\frac{s^{1-k}}{1-k}\right] e^{(n-1)\left[p(T-t)+\frac{q}{2}[T^2-t^2]\right]}\left[(1-n)\int_t^T e^{(1-n)\left[p(T-t)+\frac{q}{2}(T^2-t^2)\right]}dt + \left[\frac{1-k}{s^{1-k}}\right]^{1-n}\right]\right\}^{\frac{1}{1-n}}$$

Proof:

The aim of the study is to give an explicit solution to the investor's problem with respect to his power utility preference. The derivation of Hamilton-Jacobi-Bellman (HJB) partial differential starts with the Bellman equation;

$$J(t,S,V) = sup_{\pi}\left\{\frac{C^{1-k}}{1-k} + \frac{1}{1+\vartheta} E[J(t+\Delta t, S, V)]\right\} \tag{11}$$

The actual utility over time interval of length Δt is $\frac{C^{1-k}}{1-k}\Delta t$ and the discounting over such period is expressed as $\frac{1}{1+\vartheta\Delta t}$, $\sigma > 0$.

Therefore, the Bellman equation becomes;

$$J(t,S,V) = sup_{\pi}\left\{\frac{C^{1-k}}{1-k}\ \Delta t + \frac{1}{1+\vartheta\Delta t}E[J(t+\Delta t,S,V')]\right\} \tag{12}$$

The multiplication of (12) by $(1+\vartheta\Delta t)$ and rearranging terms obtains;

$$\vartheta J(t,\pi,v)\Delta t = sup_{\pi}\left\{\frac{C^{1-k}}{1-k}\ \Delta t\ (1+\vartheta\Delta t) + E(\Delta J)\right\} \tag{13}$$

Dividing (13) by Δt and taking limit to zero, obtains the Bellman equation;

$$\vartheta J = sup_{\pi}\left\{\frac{C^{1-k}}{1-k} + \frac{1}{dt}E(dJ)\right\} \tag{14}$$

Applying the maximum principle which states;

$$dJ = \frac{\partial J}{\partial t}dt + \frac{\partial J}{\partial s}ds + \frac{\partial J}{\partial v}dv + \frac{\partial^2 J}{\partial s\partial v}dsdv + \frac{1}{2}\left[\frac{\partial^2 J}{\partial S^2}(ds)^2 + \frac{\partial^2 J}{\partial v^2}(dv)^2\right] \tag{15}$$

But

$$\begin{aligned} < ds(t) > &= \{s[\mu dt + bs^{\gamma}dZ\ ]\}^2 \\ &= [\mu sdt + bs^{\gamma+1}dZ]^2 \\ &= b^2s^{2(\gamma+1)}dt. \end{aligned} \tag{16}$$

where

$$dt.dt\ =\ dt.dZ\ = 0, dZ.dZ = dt$$

therefore;

$$\begin{aligned} dJ = J_t dt + J_s\big[s[\mu dt + bs^{\gamma}dZ]\big] + J_v\big[[(\mu+\emptyset)-(r+\lambda+\theta)]\,\pi + [r\,\mathrm{v} - C]dt + bs^{\gamma}\pi dZ\big] \\ + J_{sv}\big[b^2s^{2(\gamma+1)}\pi\big]dt + \frac{1}{2}\big\{J_{ss}b^2s^{2(\gamma+1)}dt + J_{vv}\pi^{2b^2s^{2\gamma}dt}\big\} \\ = J_t dt + \mu sJ_s dt + bs^{\gamma+1}dZJ_s + [(\mu+\emptyset)-(r+\lambda+\theta)]\,\pi[\mathrm{rv} - C]J_v dt \\ + bs^{\gamma}\pi dZJ_v + \frac{1}{2}b^2s^{2\gamma}J_{vv}dt. \end{aligned} \tag{17}$$

Putting (17) into (14) yields;

$$\begin{aligned} \vartheta J = \frac{C^{1-k}}{1-k} + J_t + \mu sJ_s + \{[(\mu+\emptyset)-(r+\lambda+\theta)]\,\pi + [r\,\mathrm{v} - C]\}J_v + b^2s^{2(\gamma+1)}\pi J_{sv} \\ + \frac{b^2s^{2(\gamma+1)}}{2}J_{ss} + \frac{b^2s^{2\gamma}}{2}\pi^2 J_{vv} \end{aligned} \tag{18}$$

Rearranging gives;

$$\begin{aligned} \frac{C^{1-k}}{1-k} + J_t + usJ_s + \{[(\mu+\emptyset)-(r+\lambda+\theta)]\,\pi + r\,\mathrm{V} - C\}J_v + b^2s^{2(\gamma+1)}\pi J_{sv} + \frac{b^2s^{2(\gamma+1)}}{2}J_{ss} + \\ \frac{b^2\pi^2s^{2\gamma}}{2}J_{vv} - \vartheta J = 0, \end{aligned} \tag{19}$$

the required HJB equation where $J_t,\ J_s,\ J_v$ are first partial derivatives with respect to t, s and v respectively and J_{sv},J_{ss}, and J_{vv} denote second partial derivatives. The boundary condition to this problem is that $J(T,S,V) = U(V)$.

The differentiation of (19) with respect to $\pi(t)$ and $C(t)$, respectively the first order maximizing conditions for the optimal values $\pi^*(t)$ of $\pi(t)$ and $C^*(t)$ of $C(t)$ obtain the following;

$$[(\mu+\emptyset)-(r+\lambda+\theta)]J_v + b^2 s^{2(\gamma+1)} J_{sv} + \pi b^2 s^{2\gamma} J_{vv} = 0. \tag{20a}$$

which simplifies to;

$$\pi^*(t) = -\frac{[(\mu+\emptyset)-(r+\lambda+\theta)]J_v}{(bs^\gamma)^2 J_{vv}} - \frac{sJ_v}{J_{vv}}, \tag{20b}$$

for the investment in the risky asset, and

$$\frac{(1-k)C^{1-k}}{(1-k)} - J_v = 0, \tag{21a}$$

which simplifies to

$$C^* = (J_v)^{-\frac{1}{k}} = \left(\frac{1}{J_v}\right)^{\frac{1}{k}}. \tag{21b}$$

Introducing (21a) and (21b) into (19) yields;

$$\frac{1}{1-k}(J_v)^{\frac{k-1}{k}} + J_t + \mu s J_s + \left\{\beta\pi^* + rv - (J_v)^{-1/k}\right\} J_v + b^2 s^{2(\gamma+1)}\pi^* J_{sv} + \frac{b^2 s^{2\gamma}}{2} J_{ss} + \frac{b^2 s^{2\gamma}\pi^{*2}}{2} J_{vv} - \vartheta J = 0. \tag{22}$$

where $\beta = [(\mu+\emptyset)-(r+\lambda+\theta)]$,

This reduces to

$$\frac{k}{1-k}(J_v)^{\frac{k-1}{k}} + J_t + \mu s J_s + [\beta\pi^* + rv]J_v + b^2 S^{2(\gamma+1)}\pi^* J_{sv} + \frac{b^2 S^{2(\gamma+1)}}{2} J_{ss} + \frac{b^2 S^{2\gamma}\pi^{*2}}{2} J_{vv} - \vartheta J = 0. \tag{23}$$

This is a second order partial differential equation and difficult to solve.

Conjecturing a solution with the following structure;

$$\mathrm{J}(t,s,v) = \frac{v^{1-k}}{1-k} g(t,s), \tag{24a}$$

such that at terminal time T,

$$g(T,s) = 1, \tag{24b}$$

then ,

$$J_t = \frac{v^{1-k}}{1-k} g_t;\ J_s = \frac{v^{1-k}}{1-k} g_s\ , J_v = V^{-1} g$$
$$J_{vs} = v^{1-k} g_s, J_{ss} = \frac{v^{1-k}}{1-k} g_{ss}, and\ J_{vv} = -kv^{-(k+1)} g. \tag{24c}$$

Substituting (24c) into (20a) gives;

$$\pi^* = \frac{-[(\mu+\emptyset)-(r+\lambda+\theta)]v^{-k} g}{-k(bS^\gamma)^2 v^{-k-1} g} - \frac{sv^{-k} g}{-kv^{-k-1} g}. \tag{25a}$$

which simplifies to;

$$\pi^* = \frac{v}{k}\left\{\frac{[(\mu+\emptyset)-(r(t)+\lambda+\theta)]}{(bs^\gamma)^2} + s\right\}. \tag{25b}$$

Also substituting (24b) into (20b) yields;

$$C^* = (v^{-k}g)^{-\frac{1}{k}} = v\left(\frac{1}{g}\right)^{\frac{1}{k}}. \tag{26}$$

The substitution of (24b) into (23) obtains;

$$\frac{k}{1-k}[v^{-k}g]^{\frac{k-1}{k}} + \frac{v^{1-k}}{1-k}g_t + \mu s\frac{v^{1-k}}{1-k}g_s + \left\{\beta\left[\frac{v}{k}\left[\frac{\beta}{(bs^\gamma)^2} - s\right]\right] - rv\right\}v^{-k}g + \quad b^2s^{2(\gamma+1)}\left\{\frac{v}{k}\left[\frac{\beta}{(bs^\gamma)^2} - s\right]\right\}v^{-k}g_s + \frac{b^2s^{2(\gamma+1)}}{2(1-k)}v^{1-k}g_{ss} + b^2s^{2\gamma}\left\{\frac{v}{k}\left[\frac{\beta}{(bS^\gamma)^2} - s\right]\right\}^2 \times \quad (-kv^{-k-1}g) - \frac{\vartheta v^{1-k}}{1-k}g = 0. \tag{27}$$

Equation (27) simplifies to;

$$\frac{k}{1-k}v^{1-k}g^{\frac{k-1}{k}} + \frac{v^{1-k}}{1-k}g_t + \frac{\mu s v^{1-k}}{1-k}g_s + \frac{b^2s^{2(\gamma+1)}}{2}\frac{v^{-k}}{1-k}g_{ss} + v^{1-k}g\left\{\left[\frac{\beta^2}{k(bs^\gamma)^2} - \frac{s\beta}{k}\right] + r\beta\right\} - \frac{(bs^\gamma)^2}{2}v^{-k}g\left[\frac{\beta^2}{(bs^\gamma)^4} - \frac{2s\beta}{(bs^\gamma)^2} + s^2\right] - \frac{\vartheta v^{1-k}}{1-k}g = 0. \tag{28}$$

This further simplifies to;

$$kg^{\frac{k-1}{k}} + g_t + \mu s g_s + \left[bs^{(\gamma+1)}\right]^2 g_{ss} + (1-k)g\left\{\frac{\beta^2[1-(bS^\gamma)^2]}{k(bs^\gamma)^2} + \frac{s(1-k)\beta + k(1-k)r\beta - (1-k)b^2s^{2(\gamma+1)}}{k(1-k)} - k\vartheta\right\} = 0. \tag{29a}$$

Let

$$(1-k)\left\{\frac{\beta^2[1-(bS^\gamma)^2]}{k(bs^\gamma)^2} + \frac{s(1-k)\beta + k(1-k)r\beta - (1-k)b^2s^{2(\gamma+1)}}{k(1-k)} - k\vartheta\right\} = \zeta(t), \tag{29b}$$

equation (29a) becomes;

$$kg^{\frac{k-1}{k}} + g_t + \mu s g_s + \left[bs^{(\gamma+1)}\right]^2 g_{ss} + \zeta(t)g = 0. \tag{30}$$

This eliminates dependency on V, and another second order partial differential equation that is difficult to solve.

For the above reason, we conjecture that;

$$g(t,s) = \frac{s^{1-k}}{1-k}h(t); \tag{31a}$$

such that at terminal time T,

$$h(T) = \frac{1-k}{s^{1-k}}. \tag{31b}$$

This helps to eliminate the dependency on S.

From (31a) obtains the following;

$$g_t = \frac{s^{1-k}}{1-k}\frac{dh}{dt};\ g_s = s^{-k}h\ ; g_{ss} = -ks^{-k-1}h. \tag{31c}$$

Applying (31a-c) to (30) obtains

$$k\left[\frac{s^{1-k}}{1-k}h\right]^{\frac{k-1}{k}} + \frac{s^{1-k}}{1-k}\frac{dh}{dt} + \mu s(s^{-k}h) + \left(bs^{(\gamma+1)}\right)^2(-ks^{-k-1}h) + \zeta(t)\frac{s^{1-k}}{1-k}h = 0, \tag{32a}$$

which reduces to;

$$\frac{ks^{1-k}}{1-k}h^{\frac{k-1}{k}} + \frac{s^{1-k}}{1-k}h' + \mu s^{1-k}h - kb^2s^{2\gamma+2-k-1}h + \zeta(t)\frac{s^{1-k}}{1-k}h = 0, \tag{32b}$$

and further reduces to

$$h^{\frac{k-1}{k}} + h' + [(1-k)\mu - k(1-k)b^2s^{2\gamma} + \zeta(t)]h = 0, \tag{33}$$

an ordinary differential equation in h.

Rewriting (33) obtains;

$$h^{\frac{k-1}{k}} + h' + [p+qt]h = 0. \tag{34a}$$

where;

$$p + qt = [(1-k)\mu - k(1-k)b^2s^{2\gamma} + \zeta(t)] \tag{34b}$$

Let $\frac{k-1}{k} = n$, then (34a) becomes;

$$h^n + h' + [p+qt]h = 0. \tag{35}$$

Divide (35) by h^n to get;

$$1+h^{-n}h' + (p+qt)h^{1-n} = 0 \tag{36}$$

a Bernoulli equation.

Furthermore adopting a short hand variable;

$$Z=h^{1-n} \tag{37}$$

so that;

$$\frac{dZ}{dt} = (1-n)h^{-n}\frac{dh}{dt} \tag{38}$$

The preceding equation (36) can be written as;

$$1 + \frac{1}{1-n}\frac{dZ}{dt} + (p+qt)Z=0 \tag{39}$$

Rearranging (39) gives;

$$\frac{dZ}{dt} + (1-n)\,(p+qt)Z = n-1 \tag{40}$$

Equation (40) is a first order differential equation in which the variable Z has taken the place of h, and the solution can be obtained using the theorem below.

Theorem: If $Q(t)$ and $R(t)$ are continuous on the interval $I = (t, T)$, the general solution of $Z(t)$ of

$$\frac{dZ}{dt} + Q(t)Z = R(t) \text{ on } I = (t, T)$$

is given by;

$$Z(t) = e^{-\int_t^T Q(t)dt}\left[\int_t^T R(t)e^{\int_t^T Q(t)dt}\,dt + A\right], \tag{41}$$

(see [23]).

The solution to (40) above is thus;

$$Z(t) = e^{-\int_t^T (1-n)(p+qt)dt}\left[\int_t^T (n-1)e^{\int_t^T (1-n)(p+qs)ds}d\tau + A\right] \tag{42}$$

From which obtains;

$$Z(t) = e^{(n-1)\left[p(T-t)+\frac{q}{2}(T^2-t^2)\right]}\left[(n-1)\int_t^T e^{(1-n)\left[p(T-t)+\frac{q}{2}(T^2-t^2)\right]}\,d\tau + A\right]. \tag{43}$$

To obtain the value of A, the terminal condition is applied thus;

$$Z(T) = h^{1-n}(T) = \left[\frac{1-k}{s^{1-k}}\right]^{1-n} \tag{44}$$

Therefore in (42) obtains

$$Z(T) = A = \left[\frac{1-k}{s^{1-k}}\right]^{1-n} \tag{45}$$

and

$$Z(t) = e^{(n-1)\left[p(T-t)+\frac{q}{2}[T^2-t^2]\right]}\left[(1-n)\int_t^T e^{(1-n)\left[p(T-t)+\frac{q}{2}(T^2-t^2)\right]}\,dt + \left[\frac{1-k}{s^{1-k}}\right]^{1-n}\right].$$

But

$$h(t) = Z^{\frac{1}{1-n}}(t) = \left\{e^{(n-1)\left[p(T-t)+\frac{q}{2}[T^2-t^2]\right]}\left[(1-n)\int_t^T e^{(1-n)\left[p(T-t)+\frac{q}{2}(T^2-t^2)\right]}\,dt + \left[\frac{1-k}{s^{1-k}}\right]^{1-n}\right]\right\}. \tag{46}$$

Now, the optimal value function is given as;

$$J^*(t,s,v) = \left\{\left[\frac{v^{1-k}}{1-k}\right]\left[\frac{s^{1-k}}{1-k}\right]e^{(n-1)\left[p(T-t)+\frac{q}{2}[T^2-t^2]\right]}\left[(1-n)\int_t^T e^{(1-n)\left[p(T-t)+\frac{q}{2}(T^2-t^2)\right]}\,dt + \left[\frac{1-k}{s^{1-k}}\right]^{1-n}\right]\right\}^{\frac{1}{1-n}} \tag{47}$$

At the terminal time T, equation (47) becomes;

$$J^*(T,S,V) = \left[\frac{v^{1-k}}{1-k}\right]\left[\frac{s^{1-k}}{1-k}\right]\left[\frac{1-k}{s^{1-k}}\right] = \frac{v^{1-k}}{1-k} \tag{48}$$

as expected.

Applying (47) to (26), the optimal consumption is

$$C^* = \frac{V}{\frac{s^{1-k}}{1-k}\left\{e^{(n-1)\left[p(T-t)+\frac{q}{2}[T^2-t^2]\right]}\left[(1-n)\int_t^T e^{(1-n)\left[p(T-t)+\frac{q}{2}(T^2-t^2)\right]}dt+\left[\frac{1-k}{s^{1-k}}\right]^{1-n}\right]\right\}^{\frac{1}{1-n}}}. \tag{49}$$

4 Findings

Equation (25b)

$$\pi^* = \frac{v}{k}\left\{\frac{[(\mu+\emptyset)-(r+\lambda+\theta)]}{(bs^{\gamma})^2} + s\right\},$$

clearly shows that that if the sum of the drift parameter and dividend rate equals the sum of the tax rate, transaction cost rate and the rate of the return of the risk-free asset, then, the optimal investment strategy on the risky asset becomes totally dependent on the price of the risky asset and the total amount available for investment. Also, the investment strategy is horizon dependent as r, s, and v are all horizon dependent.

The optimal consumption is a function of the total amount available for investment and is also horizon dependent.

5 Conclusion

This work investigated an investor's investment consumption decision problem. It assumed the stock price followed the constant elasticity of variance, a natural extension of the geometric Brownian motion. The application of dynamic programming principles and the conjectures on elimination of variables obtained close-form solutions to the optimal investment and consumption strategies where the investor has a power utility preference and taxes, transaction costs and dividend payments are involved. It is found that if the sum of the drift parameter and dividend rate equals the sum of the tax rate, transaction cost rate and the rate of the return of the risk-free asset, then, the optimal investment strategy on the risky asset becomes totally dependent on the price of the risky asset and the total amount available for investment.

Competing Interests

Author has declared that no competing interests exist.

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