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Fixed Points for Some Multivalued Mappings in G_p-Metric Spaces

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

The aim of this work is to establish some new fixed point theorems for multivalued mappings in G_p metric space.

Keywords: Fixed point; multivalued mapping; G_p metric spaces.

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1 Introduction and Preliminaries

In 1922, Banach[1] proved a theorem about the existence and uniqueness of fixed point. Thanks to this work, many generalization theorems were introduced and generalized the Banach contraction principle in some different way.

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Nadler [2], introduced the notion of multivalued contraction mapping and proved well known Banach contraction principle. Aydi at al. [3] proved the Banach type fixed point results for set valued mapping in complete metric spaces. Matthews [4], introduced the partial metric spaces and proved a fixed point theorem on this space. After that several fixed point results have been proved in this spaces. Mustafa and Sims[5] introduced the concept of G metric spaces in the year 2006 as a generalization of the metric spaces. Recently, based on the two above metric spaces, Zand and Nezhad [6] introduced a new generalized metric spaces G_p which as a both generalization of the partial metric space and G metric spaces. Some of these works may be noted in [7, 8, 9, 10, 11, 12, 13, 14, 15].

We now reminding some fundamental definitions, notations and basic results that will be used throughout this paper.

Definition 1.1. [6] Let X be a nonempty set and let $G_p : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

$(GP1)$ $0 \leq G_p(x, x, x) \leq G_p(x, x, y) \leq G_p(x, y, z)$, all $x, y, z \in X$;

$(GP2)$ $G_p(x, y, z) = G_p(x, z, y) = G_p(y, z, x) \ldots$, (symmetry in all three variables);

$(GP3)$ $G_p(x, y, z) \leq G(x, a, a) + G_p(a, y, z) - G_p(a, a, a)$, for any $a, x, y, z \in X$, (rectangle inequality);

$(GP4)$ $x = y = z$ if $G_p(x, y, z) = G_p(x, x, x) = G_p(y, y, y) = G_p(z, z, z)$;

Then the pair (X, G_p) is called a G_p metric space.

Proposition 1.1. *[6] Let (X, G_p) be a G_p-metric space. Then for any x, y, z and $a \in X$ the following relations are true.*

(i) $G_p(x, y, z) \leq G_p(x, x, y) + G_p(x, x, z) - G_p(x, x, x)$;

(ii) $G_p(x, y, y) \leq 2G_p(x, x, y) - G_p(x, x, x)$;

(iii) $G_p(x, y, z) \leq G_p(x, a, a) + G_p(y, a, a) + G_p(z, a, a) - 2G_p(a, a, a)$;

(iv) $G_p(x, y, z) \leq G_p(x, a, z) + G_p(a, y, z) - G_p(a, a, a)$.

Definition 1.2. [6] Let (X, G_p) be a G_p-metric space and a sequence $\{x_n\}$ is called a G_p convergent to $x \in X$ if

$$\lim_{n,m\to\infty} G_p(x, x_n, x_m) = G_p(x, x, x).$$

A point $x \in X$ is said to be limit point of the sequence $\{x_n\}$ and written $x_n \to x$.

Thus if $x_n \to x$ in a G_p metric space (X, G_p), then for any $\epsilon > 0$, there exists $\ell \in \mathbb{N}$ such that $|G_p(x, x_n, x_m) - G_p(x, x, x)| < \epsilon$, for all $n, m > \ell$.

Proposition 1.2. *[6] Let (X, G_p) be a G_p-metric space, then for any sequence $\{x_n\}$ in X, the following are equivalent that*

(i) $\{x_n\}$ *is* G_p *convergent to* x;

(ii) $G_p(x_n, x_n, x) \to G_p(x, x, x)$ *as* $n \to \infty$;

(iii) $G_p(x_n, x, x) \to G_p(x, x, x)$ *as* $n \to \infty$.

Definition 1.3. [6] Let (X, G_p) be a G_p-metric space.

(i) A sequence $\{x_n\}$ is called a G_p Cauchy if and only if $\lim_{n,m\to\infty} G_p(x_n, x_m, x_m)$ exists and finite.

(ii) A G_p metric space (X, G_p) is said to be G_p complete if and only if every G_p Cauchy sequence in X is G_p convergent to $x \epsilon X$ such that
$G_p(x, x, x) = \lim_{n,m\to\infty} G_p(x_n, x_m, x_m)$.

Lemma 1.1. *[8] Let (X, G_p) be a G_p metric space. Then*

(i) If $G_p(x, y, z) = 0$ then $x = y = z$,

(ii) If $x \neq y$ then $G_p(x, y, y) > 0$.

Recently, Kaewchaeron and Kaewkhao ([16]) introduced the following concepts.

Let X be a G metric space. We shall denote $CB(X)$ the family of all nonempty closed bounded subsets of X. Let $H(.,.,.)$ be the Hausdorff G distance on $CB(X)$, i.e.,

$$H_G(A, B, C) = \max\{sup_{x\in A} G(x, B, C), sup_{x\in B} G(x, C, A), sup_{x\in C} G(x, A, B)\},$$

where

$$G(x, B, C) = d_G(x, B) + d_G(B, C) + d_G(x, C),$$
$$d_G(x, B) = \inf\{d_G(x, y), y \in B\},$$
$$d_G(A, B) = \inf\{d_G(a, b), a \in A, b \in B\}.$$

Recall that $G(x, y, C) = \inf\{G(x, y, z), z \in C\}$. A mapping $T : X \to 2^X$ is called a multivalued mapping. A point $x \in X$ is called a fixed point of T if $x \in Tx$.

Lemma 1.2. *[3] Let A and B be nonempty closed and bounded subsets of a partial metric space (X, G_p) and $h > 1$. Then, for all $a \in A$, there exists $b \in B$ such that*

$$G_p(a, b) \leq hH_{G_p}(A, B).$$

2 Main Results

Our first main result is the following theorem.

Theorem 2.1. *Let (X, G_p) be a complete G_p metric space, and $T : X \to CB(X)$ be a multivalued contractive mapping such that for all $x, y, z \in X$,*

$$H_{G_p}(Tx, Ty, Tz) \leq \alpha G_p(x, y, z) \tag{2.1}$$

where $\alpha \in (0, 1)$. Then T has a fixed point.

Proof. We define a sequence $\{x_n\}$ in X given by $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}$. Hence,

$$x_1 \in Tx_0, x_2 \in Tx_1 = T^2x_0, \ldots \tag{2.2}$$

If there exists $n_0 \in \mathbb{N}_0$ such that $x_{n_0} = x_{n_0+1}$

$$H_{G_p}(Tx_{n_0}, Tx_{n_0}, Tx_{n_0}) \leq \alpha G_p(x_{n_0}, x_{n_0}, x_{n_0})$$
$$H_{G_p}(x_{n_0+1}, x_{n_0+1}, x_{n_0+1}) \leq \alpha G_p(x_{n_0}, x_{n_0}, x_{n_0})$$

Therefore, from definition of H_{G_p}, we get $H_{G_p}(x_{n_0}, x_{n_0}, x_{n_0}) = 0$. Then, it is the clear that x_{n_0} is fixed point of T which completes the proof.

Now, let be $G_p(x_{n_0}, x_{n_0+1}, x_{n_0+1}) > 0$ with $x_{n_0} \neq x_{n_0+1}$ for every $n \in \mathbb{N}_0$. Hereby, from inequality (2.1), we have;

$$H_{G_p}(Tx_0, Tx_1, Tx_1) \leq \alpha G_p(x_0, x_1, x_1)$$
$$H_{G_p}(Tx_1, Tx_2, Tx_2) \leq \alpha G_p(x_1, x_2, x_2)$$

$$\vdots$$

$$H_{G_p}(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq \alpha G_p(x_n, x_{n+1}, x_{n+1}). \tag{2.3}$$

Let $h \in (1, \frac{1}{\alpha})$. In Lemma 1.2, we have

$$G_p(x_1, x_2, x_2) \leq hH_{G_p}(Tx_0, Tx_1, Tx_1) \leq h\alpha G_p(x_0, x_1, x_1)$$

$$\begin{aligned} G_p(x_2, x_3, x_3) \leq hH_{G_p}(Tx_1, Tx_2, Tx_2) &\leq h\alpha G_p(x_1, x_2, x_2) \\ &\leq h^2\alpha H_{G_p}(Tx_0, Tx_1, Tx_1) \\ &\leq h^2\alpha^2 G_p(x_0, x_1, x_1) \end{aligned}$$

Hence for all $n \in \mathbb{N}$;

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq hH_{G_p}(Tx_{n-1}, Tx_n, Tx_n) \leq \cdots \leq h^n\alpha^n G_p(x_0, x_1, x_1). \tag{2.4}$$

Get $k = h\alpha < 1$ for $k \in (0, 1)$. From (2.4), we write that

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq k^n G_p(x_0, x_1, x_1). \tag{2.5}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence.

$$\begin{aligned} G_p(x_n, x_{m+n}, x_{m+n}) \leq& G_p(x_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, x_{m+n}, x_{m+n}) - \\ & G_p(x_{n+1}, x_{n+1}, x_{n+1}) \\ \leq& G_p(x_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, x_{m+n}, x_{m+n}) \\ \leq& G_p(x_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, x_{n+2}, x_{n+2}) + \\ & G_p(x_{n+2}, x_{m+n}, x_{m+n}) - G_p(x_{n+2}, x_{n+2}, x_{n+2}) \\ & \vdots \\ \leq& G_p(x_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, x_{n+2}, x_{n+2}) + \\ & \cdots + G_p(x_{m+n-1}, x_{m+n}, x_{m+n}) \\ \leq& k^n G_p(x_0, x_1, x_1) + k^{n+1} G_p(x_0, x_1, x_1) + \\ & \cdots + k^{n+m-1} G_p(x_0, x_1, x_1) \\ =& \frac{k^n - k^{n+m}}{1-k} G_p(x_0, x_1, x_1). \end{aligned} \tag{2.6}$$

Where we take the limit for $m, n \to \infty$, this show that $G_p(x_n, x_{m+n}, x_{m+n}) \to 0$. Hence $\{x_n\}$ sequence is a Cauchy sequence. Also, (X, G_p) is a complete G_p metric space. There exist $u \in X$ such that $\{x_n\}$ sequence converges $u \in X$. So,

$$\lim_{n\to\infty} G_p(x_n, x_{n+1}, x_{n+1}) = \lim_{n\to\infty} G_p(x_n, u, u) = G_p(u, u, u) = 0. \tag{2.7}$$

Due to T is continuous mapping, we have

$$\lim_{n\to\infty} H_{G_p}(Tx_n, Tu, Tu) = 0. \tag{2.8}$$

So, for all $n \in \mathbb{N}$,

$$\begin{aligned} G_p(u, T_u, T_u) &\leq G_p(u, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, T_u, T_u) - G_p(x_{n+1}, x_{n+1}, x_{n+1}) \\ &\leq G_p(u, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, T_u, T_u) \\ &\leq G_p(u, x_{n+1}, x_{n+1}) + hH_{G_p}(Tx_n, Tu, Tu) \\ &\leq G_p(u, x_{n+1}, x_{n+1}) + h\alpha G_p(x_n, u, u) = G_p(u, x_{n+1}, x_{n+1}) + kG_p(x_n, u, u). \end{aligned}$$

From (2.7),

$$G_p(u, T_u, T_u) \leq 0.$$

This inequality is satisfying only $G_p(u, T_u, T_u) = 0$. Consequently, $u \in Tu$. This means that u is a fixed point of T. □

Example 2.2. *Let $X = [0, \infty)$ and define $G_p(x, y, z) = \max\{x, y, z\}$, for all $x, y, z \in X$. Then (X, G_p) is a complete G_p metric space. Also defined $T : X \to CB(X)$ a multivalued mapping, where*

$$T(x) = [0, x]$$

for all $x \in X$. Then, from Theorem 2.1 we get

$$H_{G_p}(Tx, Ty, Tz) \leq \alpha G_p(x, y, z) \tag{2.9}$$

$$H_{G_p}([0, x], [0, y], [0, z]) \leq \alpha G_p(x, y, z) \tag{2.10}$$

Let assume that

$$\begin{aligned} D_1([0, x], [0, y]) &= \sup\{d(a, [0, y]); a \in [0, x]\} \\ D_2([0, y], [0, x]) &= \sup\{d(b, [0, x]); b \in [0, y]\} \\ D_3([0, x], [0, z]) &= \sup\{d(a, [0, z]); a \in [0, x]\} \\ D_4([0, z], [0, x]) &= \sup\{d(c, [0, x]); c \in [0, z]\} \\ D_5([0, y], [0, z]) &= \sup\{d(b, [0, z]); b \in [0, y]\} \\ D_6([0, z], [0, y]) &= \sup\{d(c, [0, y]); c \in [0, z]\}. \end{aligned} \tag{2.11}$$

We write by (2.11),

$$H_{G_p}([0, x], [0, y], [0, z]) = \max\{D_1, D_2, D_3, D_4, D_5, D_6\}. \tag{2.12}$$

Suppose that $x < y < z$ then,

$$[0, x] \subset [0, y] \subset [0, z]. \tag{2.13}$$

So, for all $a \in X$ we have

$$d(a, [0, z]) \leq d(a, [0, y]) \leq d(a, [0, x]). \tag{2.14}$$

Hence,

$$\sup\{d(a, [0, z]); a \in X\} \leq \sup\{d(a, [0, y]); a \in X\} \leq \sup\{d(a, [0, x]); a \in X\} \tag{2.15}$$

Thereby, using by (2.11) and (2.15), If $a \in [0, x]$, then

$$\sup d(a, [0, z]) \leq \sup d(a, [0, y]) \Rightarrow D_3([0, x], [0, z]) \leq D_1([0, x], [0, y]).$$

If $b \in [0, y]$, then

$$\sup d(b, [0, z]) \leq \sup d(b, [0, x]) \Rightarrow D_5([0, y], [0, z]) \leq D_2([0, y], [0, x]).$$

If $c \in [0, z]$, then

$$\sup d(c, [0, y]) \leq \sup d(c, [0, x]) \Rightarrow D_6([0, z], [0, y]) \leq D_4([0, z], [0, x]).$$

From the equality of (2.12),

$$H_{G_p}([0, x], [0, y], [0, z]) = \max\{D_1, D_2, D_4\}. \tag{2.16}$$

Otherwise, from (2.10),

$$G_p(x,y,z) = \max\{x,y,z\} = z. \tag{2.17}$$

So, we have from (2.17),

$$\max\{D_1, D_2, D_4\} \le \alpha z.$$

Obviously, this is satisfying the condition of Theorem 2.1.

Mizoguchi and Takahashi proved the following theorem in [17].

Theorem 2.3. *[17] Let X be a complete metric space with metric d and let $T : X \to CB(X)$ satisfy $H(Tx,Ty) \le k(d(x,y))d(x,y)$, for all $x, y \in X$ with $x \neq y$, where k is a function of $(0,\infty)$ to $[0,1)$ such that $\limsup_{r\to t^+} k(r) < 1$ for every $t \in [0,\infty)$. Then T has a fixed point.*

We will do the proof of the following theorem, by using the proof method of Theorem 5 in [17].

Theorem 2.4. *Let (X,G_p) be a complete G_p metric space and $T : X \to CB(X)$ be a multivalued contractive mapping such that for all $x,y,z \in X$,*

$$H_{G_p}(Tx,Ty,Tz) \le k(G_p(x,y,z))G_p(x,y,z) \tag{2.18}$$

where k is a Mizoguchi-Takahashi function of $(0,\infty)$ to $[0,1)$ such that $\lim_{r\in t^+} k(r) < 1$ for every $t \in [0,\infty)$. Then T has a fixed point.

Proof. Let x_0 be arbitrary in X and we define a sequence $\{x_n\}$ in X given by $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}_0$. Hence,

$$x_1 \in Tx_0, x_2 \in Tx_1 = T^2x_0, \ldots, x_n \in T^nx_0 \ldots \tag{2.19}$$

We suppose that T has no fixed point. From the assumption for any $t > 0$ there exists positive numbers $N(t)$ and $e(t)$ such that

$$k(r) \le N(t) < 1$$

for all r with

$$t < r < t + e(t).$$

Take any $x_1 \in X$ and put $t_1 = G_p(x_1, Tx_1, Tx_1)$. In this case, when

$$G_p(x_1, Tx_1, Tx_1) < G_p(x_1, y, y)$$

for all $y \in Tx_1$, choose a positive number $\alpha(t_1)$ such that

$$\alpha(t_1) < \min\left\{e(t_1), \left(\frac{1}{N(t_1)} - 1\right)t_1\right\} \tag{2.20}$$

and

$$\varepsilon(x_1) = \min\left\{\frac{\alpha(t_1)}{t_1}, 1\right\}. \tag{2.21}$$

Hence, there exists $x_2 \in Tx_1$ such that,

$$G_p(x_1,x_2,x_2) < G_p(x_1,Tx_1,Tx_1) + \varepsilon(x_1)G_p(x_1,Tx_1,Tx_1)$$

$$= (1+\varepsilon(x_1))G_p(x_1,Tx_1,Tx_1). \tag{2.22}$$

Note that, from assumption of $x_1 \neq x_2$ by hypothesis that T has no fixed point. On the other hand

$$G_p(x_2,Tx_2,Tx_2) \le H_{G_p}(Tx_1,Tx_2,Tx_2) \le k(G_p(x_1,x_2,x_2))G_p(x_1,x_2,x_2) \tag{2.23}$$

so

$$G_p(x_1,Tx_1,Tx_1) - G_p(x_2,Tx_2,Tx_2) \geq G_p(x_1,Tx_1,Tx_1) - k(G_p(x_1,x_2,x_2))G_p(x_1,x_2,x_2)$$

and from (2.22),

$$\begin{aligned} G_p(x_1,Tx_1,Tx_1) - G_p(x_2,Tx_2,Tx_2) &> \frac{1}{1+\varepsilon(x_1)}G_p(x_1,x_2,x_2) - k(G_p(x_1,x_2,x_2))G_p(x_1,x_2,x_2) \\ &= \left(\frac{1}{1+\varepsilon(x_1)} - k(G_p(x_1,x_2,x_2))\right)G_p(x_1,x_2,x_2). \end{aligned} \tag{2.24}$$

Further than, from $t_1 = G_p(x_1,Tx_1,Tx_1)$, we get

$$t_1 = G_p(x_1,Tx_1,Tx_1) < G_p(x_1,x_2,x_2) < G_p(x_1,Tx_1,Tx_1) + \varepsilon(x_1)G_p(x_1,Tx_1,Tx_1)$$

$$\leq t_1 + \alpha(t_1) \leq t_1 + e(t_1). \tag{2.25}$$

So,

$$k(G_p(x_1,x_2,x_2)) \leq N(t_1) < 1.$$

From (2.21) and (2.20),

$$\varepsilon(x_1) \leq \frac{\alpha(t_1)}{t_1} < \frac{1}{N(t_1)} - 1 \tag{2.26}$$

$$\varepsilon(x_1) + 1 < \frac{1}{N(t_1)}$$

$$N(t_1) < \frac{1}{1+\varepsilon(x_1)}. \tag{2.27}$$

Hence,

$$\frac{1}{1+\varepsilon(x_1)} - k(G_p(x_1,x_2,x_2)) > 0 \tag{2.28}$$

In this case, since $G_p(x_1,Tx_1,Tx_1) = G_p(x_1,x_2,x_2)$ for $x_2 \in Tx_1$. We have from (2.23)

$$\begin{aligned} G_p(x_1,Tx_1,Tx_1) - G_p(x_1,x_2,x_2) &\geq G_p(x_1,Tx_1,Tx_1) - H_{G_p}(Tx_1,Tx_2,Tx_2) \\ &\geq G_p(x_1,Tx_1,Tx_1) - k(G_p(x_1,x_2,x_2))G_p(x_1,x_2,x_2) \\ &= (1 - k(G_p(x_1,x_2,x_2))G_p(x_1,x_2,x_2). \end{aligned} \tag{2.29}$$

Next, let $t_2 = G_p(x_2,Tx_2,Tx_2)$. In the case when

$$G_p(x_2,Tx_2,Tx_2) < G_p(x_2,y,y)$$

for all $y \in Tx_2$, $e(t_2)$ and $N(t_2)$, choose $\alpha(t_2)$ with

$$0 < \alpha(t_2) < \min\left\{e(t_2), \left(\frac{1}{N(t_2)} - 1\right)t_2\right\} \tag{2.30}$$

and set,

$$\varepsilon(x_2) = \min\left\{\frac{\alpha(t_2)}{t_2}, \frac{1}{2}, \frac{t_1}{t_2} - 1\right\} \tag{2.31}$$

In the same way as above, we obtain $x_3 \in Tx_2$ satisfying

$$G_p(x_2, x_3, x_3) < (1 + \varepsilon(x_2))G_p(x_2, Tx_2, Tx_2) \tag{2.32}$$

and

$$G_p(x_2, Tx_2, Tx_2) - G_p(x_3, Tx_3, Tx_3) \geq \left(\frac{1}{1+\varepsilon(x_2)} - k\left(G_p(x_2, x_3, x_3)\right)\right) G_p(x_2, x_3, x_3) > 0.$$

Since $\varepsilon(x_2) \leq \frac{t_1}{t_2} - 1$ and (2.32), then

$$G_p(x_2, x_3, x_3) < (1 + \varepsilon(x_2))\, G_p(x_2, Tx_2, Tx_2) \leq G_p(x_1, Tx_1, Tx_1) \leq G_p(x_1, x_2, x_2).$$

When $G_p(x_2, Tx_2, Tx_2) = G_p(x_2, x_3, x_3)$ for $x_3 \in Tx_2$, we have,

$$G_p(x_2, Tx_2, Tx_2) - G_p(x_3, Tx_3, Tx_3) \geq (1 - k\left(G_p(x_2, x_3, x_3)\right))\, G_p(x_2, x_3, x_3) > 0$$

and

$$G_p(x_2, x_3, x_3) = G_p(x_2, Tx_2, Tx_2) < G_p(x_1, Tx_1, Tx_1) \leq G_p(x_1, x_2, x_2).$$

Thus, for $n = 1,\ 2, \ldots$ we can inductively construct a sequence (x_n) in X with $x_{n+1} \in Tx_n$ such that $\{G_p(x_n, x_{n+1}, x_{n+1})\}_{n=1}^{\infty}$ and $\{G_p(x_n, Tx_n, Tx_n)\}_{n=1}^{\infty}$ are decreasing sequences of positive numbers and

$$\begin{aligned} &G_p(x_n, Tx_n, Tx_n) - G_p(x_{n+1}, Tx_{n+1}, Tx_{n+1}) \\ &\qquad \geq \left(\frac{1}{1+\delta(x_n)} - k\left(G_p(x_n, x_{n+1}, x_{n+1})\right)\right) G_p(x_n, x_{n+1}, x_{n+1}) \end{aligned} \tag{2.33}$$

where $\delta(x_n)$ is real numbers with

$$0 \leq \delta(x_n) \leq \frac{1}{n},\ \ (n = 1, 2, \ldots) \tag{2.34}$$

So, the sequence $\{G_p(x_n, x_{n+1}, x_{n+1})\}$ of positive real numbers converges to nonnegative number. By the assumption of the theorem,

$$\limsup_{n\to\infty}(G_p(x_n, x_{n+1}, x_{n+1})) < 1.$$

Let choose,

$$\alpha_n = \frac{1}{1+\delta(x_n)} - k(G_p(x_n, x_{n+1}, x_{n+1})), \quad (n = 1, 2, \ldots),$$

we have

$$\liminf_{n\to\infty} \alpha_n \geq \lim_{n\to\infty} \frac{1}{1+\delta(x_n)} - \limsup_{n\to\infty}\ k(G_p(x_n, x_{n+1}, x_{n+1})) > 0 \tag{2.35}$$

and there exists $\beta > 0$ such that

$$G_p(x_n, Tx_n, Tx_n) - G_p(x_{n+1}, Tx_{n+1}, Tx_{n+1}) \geq \beta G_p(x_n, x_{n+1}, x_{n+1}) \tag{2.36}$$

for large enough n. Also that, the decreasing sequence $\{G_p(x_n, Tx_n, Tx_n)\}$ of positive real numbers is convergent, we have

$$\begin{aligned} G_p(x_n, x_m, x_m) &\leq \sum_{j=n}^{m-1} G_p(x_j, x_{j+1}, x_{j+1}) \\ &< \frac{1}{\beta} \sum_{j=n}^{m-1} \{G_p(x_j, Tx_j, Tx_j) - G_p(x_{j+1}, Tx_{j+1}, Tx_{j+1})\} \\ &= \frac{1}{\beta}\{G_p(x_n, Tx_n, Tx_n) - G_p(x_m, Tx_m, Tx_m)\} \to 0. \end{aligned}$$

as $n, m \to \infty$ and hence the sequence $\{x_n\}$ in X convergence to $x_0 \in X$. If $x_0 \neq x_n$ then

$$H_{G_p}(Tx_0, Tx_n, Tx_n) \leq k(G_p(x_0, x_n, x_n))G_p(x_0, x_n, x_n) \tag{2.37}$$

and if $x_0 = x_n$ then

$$H_{G_p}(Tx_0, Tx_n, Tx_n) \leq G_p(x_0, x_n, x_n) \tag{2.38}$$

So, $x_0 \in Tx_0$ from Lemma 2 of [15]. This shows that T has a fixed point. □

Example 2.5. *Let $X = [0, \infty)$ and defined by (X, G_p) be a complete G_p metric space where*

$$G_p(x, y, z) = \max\{x, y, z\} \tag{2.39}$$

for all $x, y, z \in X$. Also defined $T : X \to CB(X)$ a multivalued mapping where

$$T(x) = \begin{cases} [-1, 1], & x \in (-\infty, 0] \\ [0, x], & x \in (0, \infty) \end{cases} \tag{2.40}$$

and $k : [0, \infty) \to [0, 1)$ be a function such that

$$k(t) = \begin{cases} 0, & t \in [0, 1) \\ \frac{1}{2t}, & t \in [1, \infty) \end{cases} \tag{2.41}$$

for every $t \in [0, \infty)$ which $\lim_{r \in t^+} k(r) < 1$. Then by using the theorem

$$H_{G_p}(Tx, Ty, Tz) \leq k(G_p(x, y, z))G_p(x, y, z).$$

If $x, y, z \in (-\infty, 0]$, we get,

$$\begin{aligned} H_{G_p}(Tx, Ty, Tz) &= H_{G_p}([-1, 1], [-1, 1], [-1, 1]) \\ &= \max\{D([-1, 1], [-1, 1])\}, \end{aligned}$$

and

$$D([-1, 1], [-1, 1]) = \sup_{a \in [-1,1]} d(a, [-1, 1]) = 0.$$

So,

$$0 \leq k(G_p(x, y, z))G_p(x, y, z).$$

from (2.39), we have that

$$G_p(x, y, z) = \max\{x, y, z\} = 0, x, y, z \in (-\infty, 0]$$

This is satisfying the Theorem 2.4. On the other hand, $x, y, z \in (0, \infty)$, we get,

$$H_{G_p}(Tx, Ty, Tz) = H_{G_p}([0, x], [0, y], [0, z]) \tag{2.42}$$

from the assumption (2.11) of the Example 2.2 , we write,

$$H_{G_p}([0, x], [0, y], [0, z]) = \max\{D_1, D_2, D_3, D_4, D_5, D_6\}. \tag{2.43}$$

Let be $x < y < z$ such that we get

$$[0, x] \subset [0, y] \subset [0, z]. \tag{2.44}$$

Then,

$$d(a,[0,z]) \leq d(a,[0,y]) \leq d(a,[0,x]). \qquad (\forall a \in X) \tag{2.45}$$

Hence,

$$\sup\{d(a,[0,z]); a \in X\} \leq \sup\{d(a,[0,y]); a \in X\} \leq \{d(a,[0,x]); a \in X\}. \tag{2.46}$$

Thereby,
If $a \in [0,x]$*, then,*

$$\sup d(a,[0,z]) \leq \sup d(a,[0,y]) \Rightarrow D_3([0,x],[0,z]) \leq D_1([0,x],[0,y]).$$

If $b \in [0,y]$*, then,*

$$\sup d(b,[0,z]) \leq \sup d(b,[0,x]) \Rightarrow D_5([0,y],[0,z]) \leq D_2([0,y],[0,x]).$$

If $c \in [0,z]$*, then,*

$$\sup d(c,[0,y]) \leq \sup d(c,[0,x]) \Rightarrow D_6([0,z],[0,y]) \leq D_4([0,z],[0,x]).$$

From the equality, we get

$$H_{G_p}(Tx,Ty,Tz) = \max\{D_1, D_2, D_4\}.$$

Otherwise, from (2.39),

$$G_p(x,y,z) = \max\{x,y,z\} = z.$$

We have,

$$\max\{D_1, D_2, D_4\} \leq k(z)z$$

From $z \in (0,\infty)$*, we have two cases. First case*
If $z \in (0,1)$ *then,*

$$\max\{D_1, D_2, D_4\} \leq 0.$$

This is clearly that is satisfying. Other case, $z \in [1,\infty)$*, then,*

$$\max\{D_1, D_2, D_4\} \leq \frac{1}{2z}z$$

$$\max\{D_1, D_2, D_4\} \leq \frac{1}{2}.$$

Hence all the conditions of the Theorem 2.4 are satisfied.

3 Conclusion

In this paper, we gave some new fixed point theorems for multivalued mappings in G_p metric space. We hope that our study contributes to the development of these results by other researchers.

Disclaimer

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Competing Interests

Authors have declared that no competing interests exist.

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