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Derivation of a Scalable Solution for the Problem of Factoring an n-bit Integer

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Authors' contributions

This work was carried out in collaboration among the three authors. Author AMR designed the study, performed the analysis, solved the examples and wrote the preliminary manuscript. Author SSZ managed the literature search and drew the figures. Author ASB contributed to the analysis and solutions of the examples and provided useful insight about the interrelationships among problems of different sizes and various solution approaches. All authors read and approved the final manuscript.

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Abstract

The problem of integer factorization is ubiquitous in scientific and engineering applications including the challenging task of cryptanalysis. This problem is intractable but might admit real-time hardware solutions for small bit sizes. This paper suggests manual and automated scalable solutions for integer factorization based on equation solving over big Boolean algebras. The manual solution is illustrated over a form of 8-variable Karnaugh maps that is highly regular and modular. This solution covers the problem of 6 bits, which includes the problems of 5, 4, and 3 bits as special cases. Moreover, the automated solution is implemented, and subsequently its results are presented and discussed briefly. These results show the notorious evolution of the temporal and spatial complexities as the number of input bits increases. Based on the automated solution, the largest possible hardware circuit obtained via the automated solution is to be constructed, verified and tested. Such a hardware implementation (e.g., FPGA implementation) could serve as a ready real-time look-up solution not only of the pertinent problem but also of all smaller problems.

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1 Introduction

The problem of integer factorization is ubiquitous in scientific applications, and it is particularly prominent in the cryptanalysis of the RSA cryptosystem [1-4]. This problem is intractable, and none of the many sophisticated algorithms for solving it has an under-exponential temporal complexity. Currently, there are many attempts to handle this problem in real time *via* hardware solutions. Some of these attempts are based on the extension of propositional logic to higher-order logic such as first-order predicate logic. These attempts involve Boolean functional synthesis and the utilization of Skolem functions [5-9]. Other attempts require the enlargement of two-valued Boolean algebras to a 'big' Boolean algebra [10-12], an approach to be pursued further herein. Other notable approaches for the hardware solution of integer factorization are also available, and continuous innovations in such approaches are being offered with no end in sight [13-28].

Our present paper aims to obtain a hardware solution for integer factorization that is based on solving Boolean equations over 'big' Boolean algebras [29-52]. However, our solution herein has three advantages over earlier solutions, namely: (a) it is general and scalable, (b) it is automated, and (c) it is realizable via current hardware technologies such as that of FPGA. Of course, scalability is not absolute. It will reach a limit due the bottleneck imposed by time and memory limitations. The solution is pictorially insightful thanks to efficient utilization of modern versions of the Karnaugh map [53-62].

The organization of the rest of this paper is as follows. Section 2 modifies our earlier hardware solution for integer factorization using Boolean-equation solving [10-12]. The modifications introduced make this solution scalable and consequently readily amenable for clear and straightforward algorithmic formulation. Section 3 solves the (6, 5, 3) factorization problem, in which a 6-bit integer **X** is factored as a product of a 5-bit integer **Y** and a 3-bit integer $\mathbf{Z}$. We solve this problem with the aid of a special form of the 8-variable Karnaugh map (taken from [62, 63]) that has better regularity and modularity than other forms of the 8-variable map such as the ones used in [11, 64-68]. Section 4 outlines our algorithmic implementation of integer factorization, with a code listing in Matlab given in appendix A. Section 4 also reports typical results obtained by running the given code. Section 5 concludes the paper.

2 General Scalable Formulation

We have earlier discussed the problem of factoring an integer **X** into a product $\mathbf{Y} * \mathbf{Z}$ of the two integers **Y** and **Z** subject to the constraints ($\mathbf{Y} \geq \mathbf{Z}$) and ($\mathbf{Z} \geq \mathbf{1}$) [10, 11]. For an even bit size of **X,** say $2n$, the bit size of **Y** and **Z** are $(2n-1)$ and n, respectively. However, when we multiply **Y** and **Z** of such sizes, we produce an **X** of bit size $(2n-1)+n=3n-1$. In our earlier work [10, 11], we have forbidden bit sizes of **X** exceeding $2n$, a feature that did not allow full scalability for our solution. In our present paper, we will *temporarily* allow bit sizes of **X** to be $(3n-1)$, thereby securing full scalability for our solution, in the sense that our solution for the $2n$ problem includes solutions for all smaller valid problem (down to a bit size of 3 for **X**) as special cases. To solve our current problem, in which the triad (**X**, **Y**, **Z**) has bit sizes $(2n, 2n-1, n)$, we need a multiplication table for the $(2n-1)-bit$ **Y** with the $n-bit$ **Z**, and allow entries of the table $T(\mathbf{Y}, \mathbf{Z}) = \mathbf{Y} * \mathbf{Z}$ to be each of $(3n-1)$ bits. For convenience, we arrange the input domain of this table to render it a Karnaugh-map layout, *i.e.,* to employ a reflected binary coding (grey coding) for each of its two dimensions.

The multiplication table constitutes the initial specification for our problem, namely

$$g_0(\mathbf{Y}, \mathbf{Z}) = 1 \qquad (1)$$

Where

$$g_0 = \wedge_{i=0}^{(3n-2)} (\mathrm{X}_i \,\mathcal{O}\, \mathrm{T}_i(\mathbf{Y},\mathbf{Z})) = \wedge_{i=0}^{(3n-2)} \mathrm{X}_i^{\mathrm{T}_i(\mathbf{Y},\mathbf{Z})} \tag{2}$$

is an ANDing over all the $(3n-1)$ bits i of $\mathbf{X}$. The XNOR function $\mathrm{X}_i \,\mathcal{O}\, \mathrm{T}_i(\mathbf{Y},\mathbf{Z})$ equals the complemented literal $\overline{\mathrm{X}}_i$ when $\mathrm{T}_i(\mathbf{Y},\mathbf{Z}) = 0$ and equals the un-complemented literal X_i when $\mathrm{T}_i(\mathbf{Y},\mathbf{Z}) = 1$. The initial function $g_0(\mathbf{Y},\mathbf{Z})$ is now replaced by the final-specification function (equated to 1)

$$g(\mathbf{Y},\mathbf{Z}) = g_0(\mathbf{Y},\mathbf{Z})\, \mathrm{I}(\mathbf{Z} > 1)\, \mathrm{I}(\mathbf{Y} \geq \mathbf{Z}) \tag{3}$$

where the symbol I(event) denotes the Boolean indicator for that event, namely

$$\mathrm{I(event)} = \begin{cases} 1 & \textit{if the event occurs} \\ 0 & \textit{if the event does not occur} \end{cases} \tag{4}$$

The two functions $g_0(\mathbf{Y},\mathbf{Z})$ and $g(\mathbf{Y},\mathbf{Z})$ are defined as $g_0\colon \mathrm{B}^{3n-1} \to \mathrm{B}$, and $g\colon \mathrm{B}^{3n-1} \to \mathrm{B}$ where B is the ‘big’ Boolean algebra $\mathrm{B} = \mathrm{FB}(\mathbf{X})$, *i.e.,* it is the free Boolean algebra with $(3n-1)$ generators $\mathrm{X}_{(3n-2)}, \mathrm{X}_{(3n-3)} \dots, \mathrm{X}_1, and\ \mathrm{X}_0$. This Boolean algebra is of $\mathrm{K} = 2^{(3n-1)}$ atoms and 2^{K} elements. The Boolean equation (1) is now solved by constructing the auxiliary function $\mathrm{G}(\mathbf{Y},\mathbf{Z},\mathbf{p})$ according to the rules given or demonstrated in [10, 11, 40, 50-52]. It is straightforward to use $\mathrm{G}(\mathbf{Y},\mathbf{Z},\mathbf{p})$ to deduce the value of $\mathrm{Y}_i(\mathbf{X},\mathbf{p}), 0 \leq i \leq (2n-2), and\ \mathrm{Z}_i, 0 \leq i \leq (n-1)$ subject to a consistency condition (to be derived also from $\mathrm{G}(\mathbf{Y},\mathbf{Z},\mathbf{p})$). We do not need to develop $\mathrm{G}(\mathbf{Y},\mathbf{Z},\mathbf{p})$ fully, as we do not need to find the parameters associated with atoms that have non-zero $\mathrm{X}_i, 2n \leq i < (3n-1)$. This means that we work with the first 2^{2n} atoms out of the $2^{(3n-1)}$ atoms. Our solutions will not involve $\mathrm{X}_i, 2n \leq i < (3n-1)$ (which are set identically zero) and will involve only the first $2n$ bits $(\mathrm{X}_i, 0 \leq i < 2n)$.

The aforementioned solution for the $(2n, 2n-1, n)$ problem includes all valid smaller problems down to the (3, 2, 2) problem as special cases. The next problem is the $(2n-1, 2n-2, n)$, which involves **X** of an odd rather than even bit size and its Karnaugh maps are of sizes that are *one half* those of the preceding $(2n, 2n-1, n)$ problem. For this latter problem, **X** has an initial odd bit size of $(3n-2)$, but only $(2n-1)$ bits are retained at the end. The problem next to this problem is the $(2n-2, 2n-3, n-1)$ problem. This problem involves **X** of an even bit size, and its Karnaugh maps are of sizes that are one *quarter* those of the preceding problem. For this problem **X** has a bit size of $(3n-4)$ at the outset, which reduces to $(2n-2)$ at the end.

3 Solution of the (6, 5, 3) Problem

Fig. 1 shows a decimal-entered multiplication table for the two integers $\mathbf{Y} = (\mathrm{Y}_4, \mathrm{Y}_3, \mathrm{Y}_2, \mathrm{Y}_1, \mathrm{Y}_0)$ and $\mathbf{Z} = (\mathrm{Z}_2, \mathrm{Z}_1, \mathrm{Z}_0)$. The table is cast in an 8-variable Karnaugh map layout. For convenience, we depict every map column by two decimal values of $\mathbf{Y} = (2^4\mathrm{Y}_4 + 2^3\mathrm{Y}_3 + 2^2\mathrm{Y}_2 + 2^1\mathrm{Y}_1 + 2^0\mathrm{Y}_0)$ in its cells $(0 \leq \mathbf{Y} \leq 31)$. The first value is $(2^4\mathrm{Y}_4 + 2^3\mathrm{Y}_3 + 2^2\mathrm{Y}_2 + 2^1\mathrm{Y}_1)$. It is determined by the four horizontal variables of the map, and is valid for cells in which $\mathrm{Y}_0 = 0$ (cells outside the Y_0 domain). The second value is equal to the previous value augmented by 1 and is valid for cells in which $\mathrm{Y}_0 = 1$ (cells inside the Y_0 domain, highlighted by green shading lines). We also depict every two consecutive map rows by a single value of $\mathbf{Z}(0 \leq \mathbf{Z} \leq 7)$, where $\mathbf{Z} = 2^2\mathrm{Z}_2 + 2^1\mathrm{Z}_1 + 2^0\mathrm{Z}_0$. Note that the *left half* of the map in Fig. 1 describes the next smaller problem, the (5, 4, 3) problem. The *top left quarter* of this half depicts the next smaller problem, the (4, 3, 2) problem. Finally, the smallest valid problem, the (3, 2, 2) problem, is represented by, again, the left half of the previous problem. We use bold boundaries and various shadings to distinguish the smaller maps used for the smaller problems in Fig. 1.

Table 1. Orthonormal tags used with composite integers less than 64 with multiple factorizations. A thick line denotes the lower boundary for the n-bit problem, where *n* = 3, 4, 5, and 6, respectively

Integer	Corresponding atom (only atoms with $\bar{X}_7\bar{X}_6$ are retained)	Multiplicity of non-trivial factorizations	Set of orthonormal tags
12	$\bar{X}_7\bar{X}_6\bar{X}_5\bar{X}_4X_3X_2\bar{X}_1\bar{X}_0$	2	$\{p_1, \bar{p}_1\}$
16	$\bar{X}_7\bar{X}_6\bar{X}_5X_4\bar{X}_3\bar{X}_2\bar{X}_1\bar{X}_0$	2	$\{p_2, \bar{p}_2\}$
18	$\bar{X}_7\bar{X}_6\bar{X}_5X_4\bar{X}_3\bar{X}_2X_1\bar{X}_0$	2	$\{p_3, \bar{p}_3\}$
20	$\bar{X}_7\bar{X}_6\bar{X}_5X_4\bar{X}_3X_2\bar{X}_1\bar{X}_0$	2	$\{p_4, \bar{p}_4\}$
24	$\bar{X}_7\bar{X}_6\bar{X}_5X_4X_3\bar{X}_2\bar{X}_1\bar{X}_0$	3	$\{p_5p_6, p_5\bar{p}_6, \bar{p}_5\}$
28	$\bar{X}_7\bar{X}_6\bar{X}_5X_4X_3X_2\bar{X}_1\bar{X}_0$	2	$\{p_7, \bar{p}_7\}$
30	$\bar{X}_7\bar{X}_6\bar{X}_5X_4X_3X_2X_1\bar{X}_0$	3	$\{p_8p_9, p_8\bar{p}_9, \bar{p}_8\}$
32	$\bar{X}_7\bar{X}_6X_5\bar{X}_4\bar{X}_3\bar{X}_2\bar{X}_1\bar{X}_0$	2	$\{p_{10}, \bar{p}_{10}\}$
36	$\bar{X}_7\bar{X}_6X_5\bar{X}_4\bar{X}_3X_2\bar{X}_1\bar{X}_0$	4	$\{p_{11}p_{12}, p_{11}\bar{p}_{12}, \bar{p}_{11}p_{12}, \bar{p}_{11}\bar{p}_{12}\}$
40	$\bar{X}_7\bar{X}_6X_5\bar{X}_4X_3\bar{X}_2\bar{X}_1\bar{X}_0$	3	$\{p_{13}p_{14}, p_{13}\bar{p}_{14}, \bar{p}_{13}\}$
42	$\bar{X}_7\bar{X}_6X_5\bar{X}_4X_3\bar{X}_2X_1\bar{X}_0$	3	$\{p_{15}p_{16}, p_{15}\bar{p}_{16}, \bar{p}_{15}\}$
44	$\bar{X}_7\bar{X}_6X_5\bar{X}_4X_3X_2\bar{X}_1\bar{X}_0$	2	$\{p_{17}, \bar{p}_{17}\}$
45	$\bar{X}_7\bar{X}_6X_5\bar{X}_4X_3X_2\bar{X}_1X_0$	2	$\{p_{18}, \bar{p}_{18}\}$
48	$\bar{X}_7\bar{X}_6X_5X_4\bar{X}_3\bar{X}_2\bar{X}_1\bar{X}_0$	4	$\{p_{19}p_{20}, p_{19}\bar{p}_{20}, \bar{p}_{19}p_{20}, \bar{p}_{19}\bar{p}_{20}\}$
50	$\bar{X}_7\bar{X}_6X_5X_4\bar{X}_3\bar{X}_2X_1\bar{X}_0$	2	$\{p_{21}, \bar{p}_{21}\}$
52	$\bar{X}_7\bar{X}_6X_5X_4\bar{X}_3X_2\bar{X}_1\bar{X}_0$	2	$\{p_{22}, \bar{p}_{22}\}$
54	$\bar{X}_7\bar{X}_6X_5X_4\bar{X}_3X_2X_1\bar{X}_0$	3	$\{p_{23}p_{24}, p_{23}\bar{p}_{24}, \bar{p}_{23}\}$
56	$\bar{X}_7\bar{X}_6X_5X_4X_3\bar{X}_2\bar{X}_1\bar{X}_0$	3	$\{p_{25}p_{26}, p_{25}\bar{p}_{26}, \bar{p}_{25}\}$
60	$\bar{X}_7\bar{X}_6X_5X_4X_3X_2\bar{X}_1\bar{X}_0$	5	$\{p_{27}p_{28}p_{29}, p_{27}p_{28}\bar{p}_{29}, p_{27}\bar{p}_{28}p_{29}, p_{27}\bar{p}_{28}\bar{p}_{29}, \bar{p}_{27}\}$
63	$\bar{X}_7\bar{X}_6X_5X_4X_3X_2X_1X_0$	2	$\{p_{30}, \bar{p}_{30}\}$

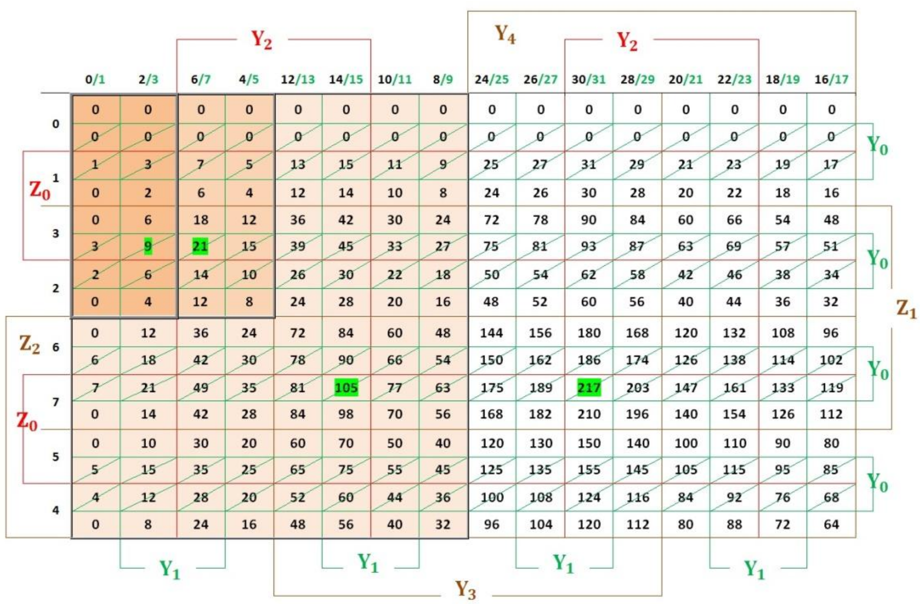
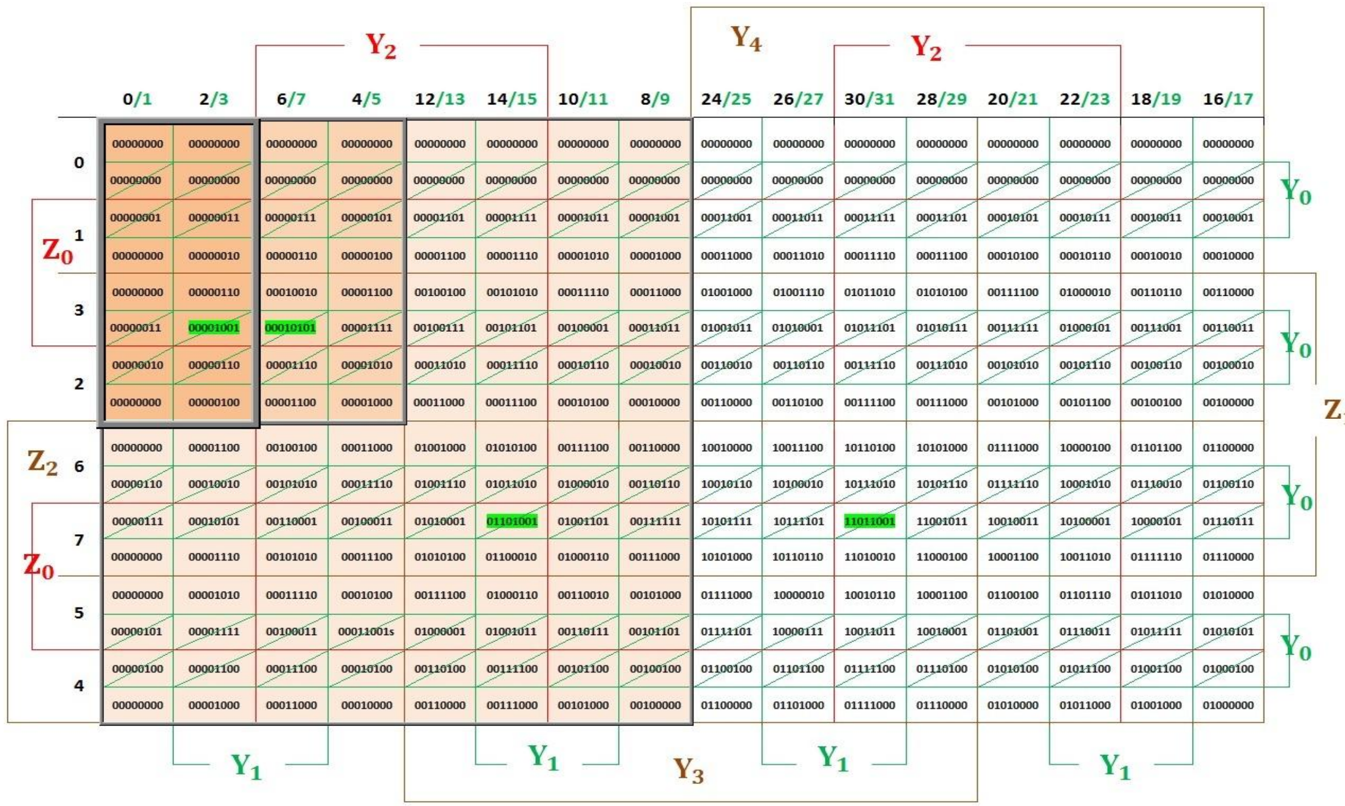


Fig. 1. The multiplication table for the (6, 5, 3) problem including its (5, 4, 3), (4, 3, 2), and (3, 2, 2) sub-problems

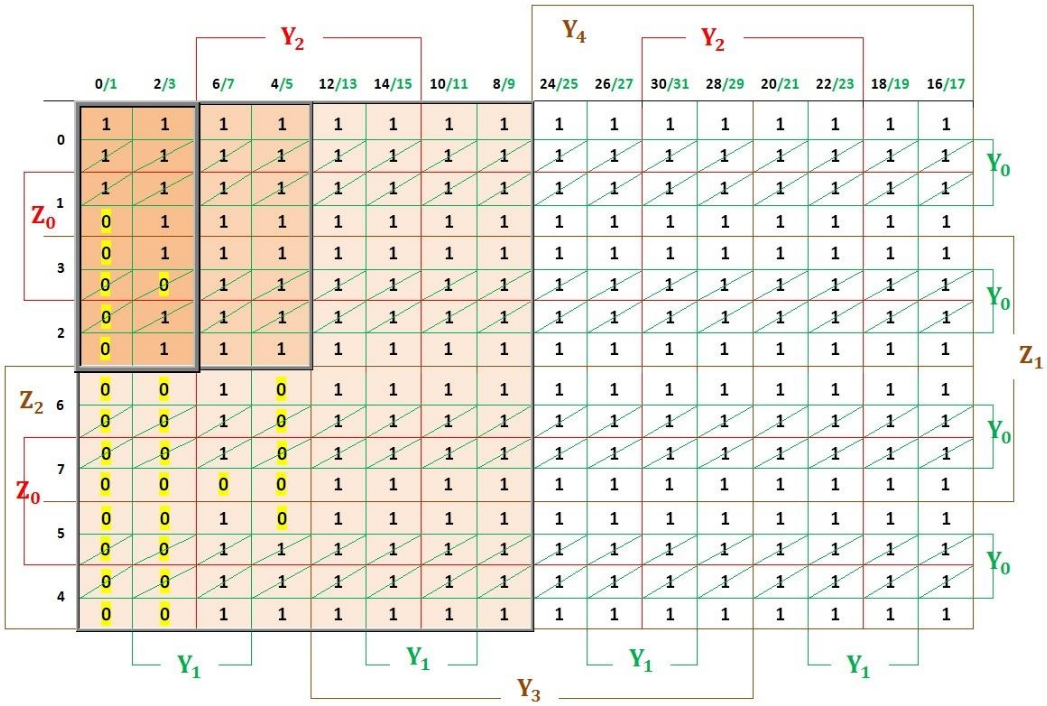
The table has the layout of an 8-variable Karnaugh map, where every column is topped by two possible decimal values for $(Y_4Y_3Y_2Y_1Y_0)$ and every two consecutive rows are labelled by a common decimal value for $(Z_2Z_1Z_0)$. Map entries represent the product $(Y_4Y_3Y_2Y_1Y_0) * (Z_2Z_1Z_0)$ in decimal. For convenience, composite numbers less than 64 having multiple factorizations are highlighted in red



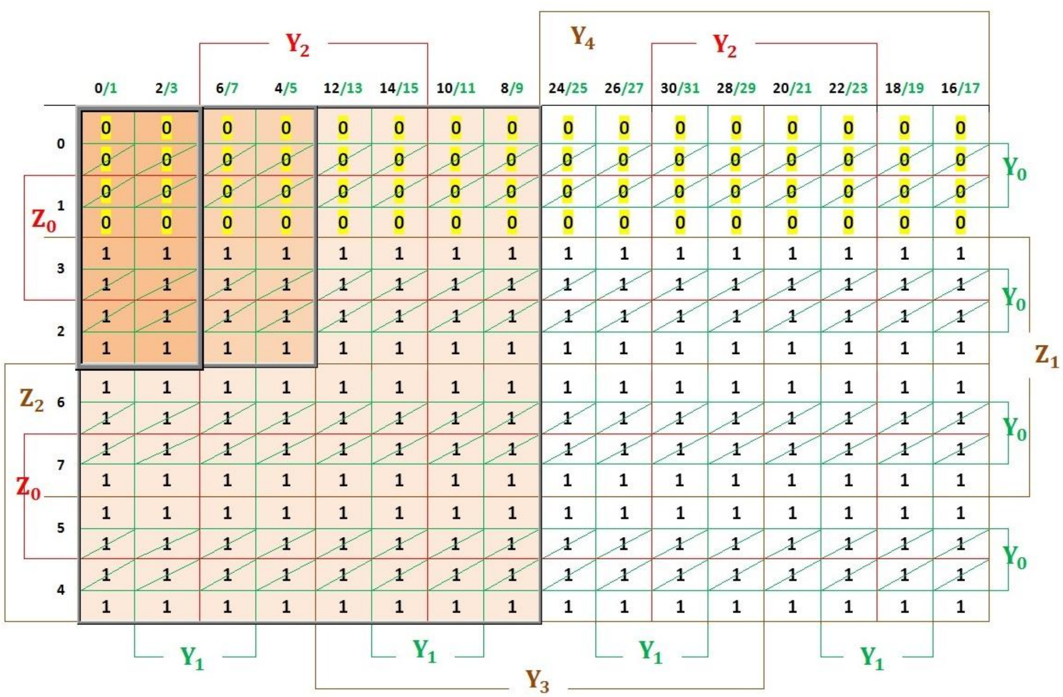
$\boldsymbol{g_0}(\mathbf{Y}, \mathbf{Z})$

Fig. 2. The multiplication table or map for the (6, 5, 3) problem including its (5, 4, 3), (4, 3, 2) and (3, 2, 2) sub-problems

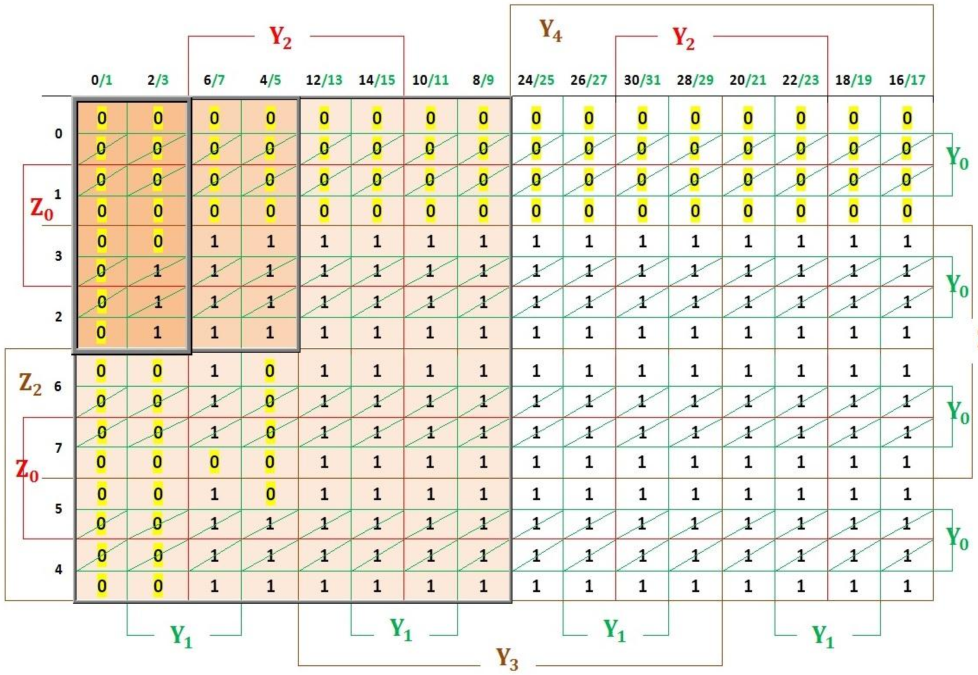
Map entries represent the product $(Y_4Y_3Y_2Y_1Y_0) * (Z_2Z_1Z_0)$ in binary (8 bits). This map also represents the initial-specification function $g_0(Y, Z)$ with binary strings understood to depict corresponding atoms. The largest entries in the map and its sub-maps (highlighted in green) are 217, 105, 21, and 9, and represent, 11011001, 01101001, 00010101, and 00001001 or, equivalently; $X_7X_6\bar{X}_5X_4X_3\bar{X}_2\bar{X}_1X_0$, $\bar{X}_7X_6X_5\bar{X}_4X_3\bar{X}_2\bar{X}_1X_0$, $\bar{X}_7\bar{X}_6\bar{X}_5X_4\bar{X}_3X_2\bar{X}_1X_0$, and $\bar{X}_7\bar{X}_6\bar{X}_5\bar{X}_4X_3\bar{X}_2\bar{X}_1X_0$.



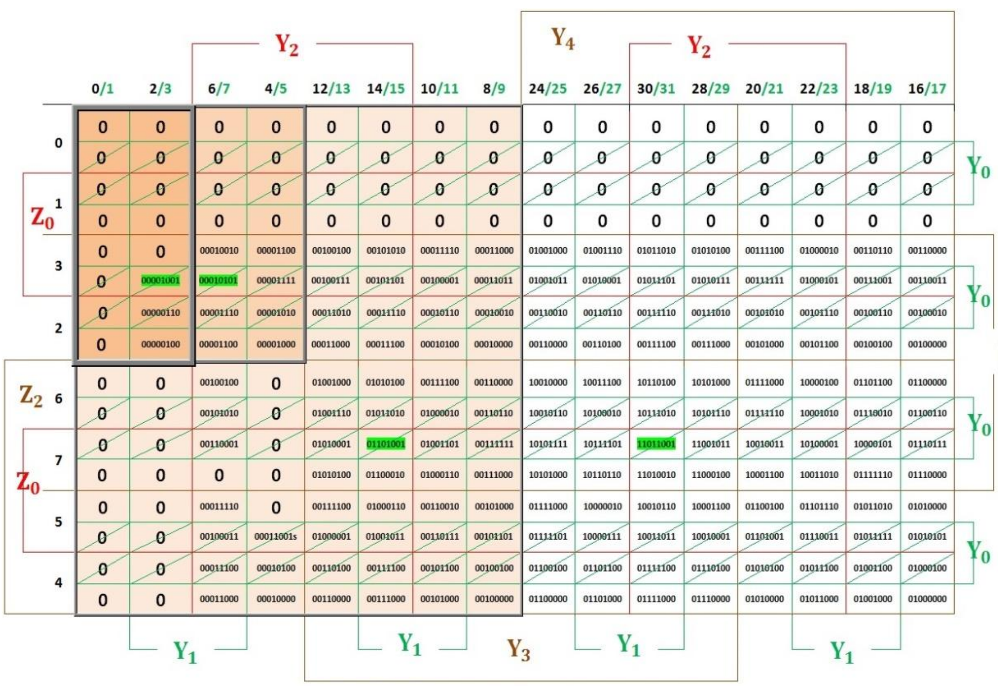
(a) $\mathbf{I(Y \geq Z)}$



(b) $\mathbf{I(Z > 1)}$



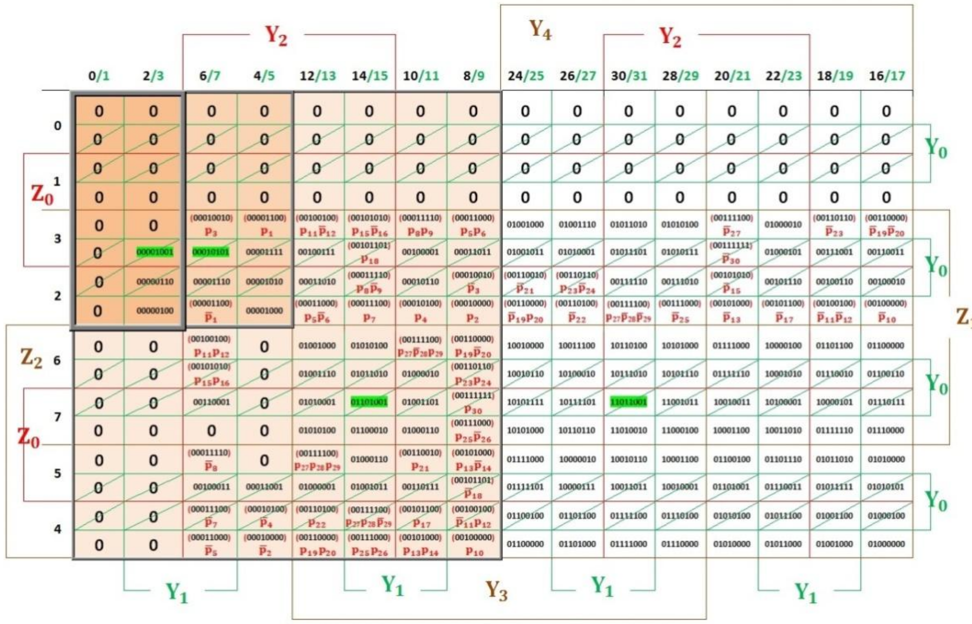
(c) $\mathbf{I(Z > 1)\, I(Y \geq Z)}$



(d) $\boldsymbol{g}(\mathbf{Y}, \mathbf{Z})$

Fig. 3. Evolution of the specification function $\boldsymbol{g}(\mathbf{Y}, \mathbf{Z})$

A non-zero entry in the $g(Y, Z)$ map symbolizes a single atom of the 256 atoms of $FB(X_7, X_6, X_5, X_4, X_3, X_2, X_1, X_0)$. For example, the entry 00111000 (binary for 56) denotes $(\bar{X}_7 \bar{X}_6 X_5 X_4 X_3 \bar{X}_2 \bar{X}_1 \bar{X}_0)$



Curtailed $\mathbf{G}(\mathbf{Y}, \mathbf{Z}, \mathbf{p})$

Fig. 4. The curtailed auxiliary function for the (6, 5, 3) problem

Fig. 2 is a replica of Fig .1, in which the entries $\mathbf{X}$ are converted from decimal to 8-bit binary representation. The figure can also be understood to represent the initial-specification function $g_0(\mathbf{Y}, \mathbf{Z})$ provided every string of bits $(T_7T_6T_5T_4T_3T_2T_1T_0)$ is understood to indicate the corresponding atom $(X_7^{T_7}X_6^{T_6}X_5^{T_5}X_4^{T_4}X_3^{T_3}X_2^{T_2}X_1^{T_1}X_0^{T_0})$. For example, the cell corresponding to $\mathbf{Y} = 31$ and $\mathbf{Z} = 7$ has an entry of $31 * 7 = (217)_{10} = (11011001)_2$ which is understood to represent $X_7X_6\overline{X}_5X_4X_3\overline{X}_2\overline{X}_1X_0$.

Fig. 3 describes the evolution of $g_0(\mathbf{Y}, \mathbf{Z})$ into $g(\mathbf{Y}, \mathbf{Z})$. Fig. 3(a) represents the indicator $I(\mathbf{Y} \geq \mathbf{Z})$, while Fig. 3(b) represents the indicator $I(\mathbf{Z} > 1)$. Fig. 3(c) is the product of Fig. 2, Fig. 3(a), and Fig. 3(b) and hence represents $g(\mathbf{Y}, \mathbf{Z})$. Fig. 4 represents the auxiliary function $G(\mathbf{Y}, \mathbf{Z}, \mathbf{p})$, curtailed to include parameters for the pertinent atoms among the 64 atoms in which $X_7 = X_6 = 0$. The transition from the specification function $g(\mathbf{Y}, \mathbf{Z})$ to the auxiliary function $G(\mathbf{Y}, \mathbf{Z}, \mathbf{p})$ is achieved according to the procedure in [10-12,40]. It is accomplished with the aid of Table 1, which identifies sets of orthonormal tags to be associated with atoms of multiple appearances in the map of $g(\mathbf{Y}, \mathbf{Z})$ in Fig. 3(d). Atoms beyond the initial 64 atoms are ignored in writing the final solution and its consistency condition, which turn out to be exactly the same as in Rushdi et al. [11]. For brevity, we have not repeated the logical expressions for the outputs $\mathbf{Z}$ and the consistency condition in this paper.

4 Algorithmic Implementation of Integer Factorization

Our manual work on integer factorization has to be stopped at the (6, 5, 3) problem. The next larger problem (namely, the (7, 6, 4)) has an input domain of 6+4=10 variables and would be considerably difficult (albeit, not totally impossible) for a Karnaugh-map treatment. Our solution procedure, on the other hand, is algorithmic in nature, and is amenable to coding as a computer program. In fact, we did write such a program in Matlab based on the knowledge accumulated throughout the manual solution of small problems. The program correctness was verified for the four problems solved manually, namely, the (3, 2, 2), (4, 3, 2), (5, 4, 3) and (6, 5, 3) problems. Fig. 5 outlines the scheme of the manual and automated approaches used herein. The figure also indicates directions for forthcoming work.

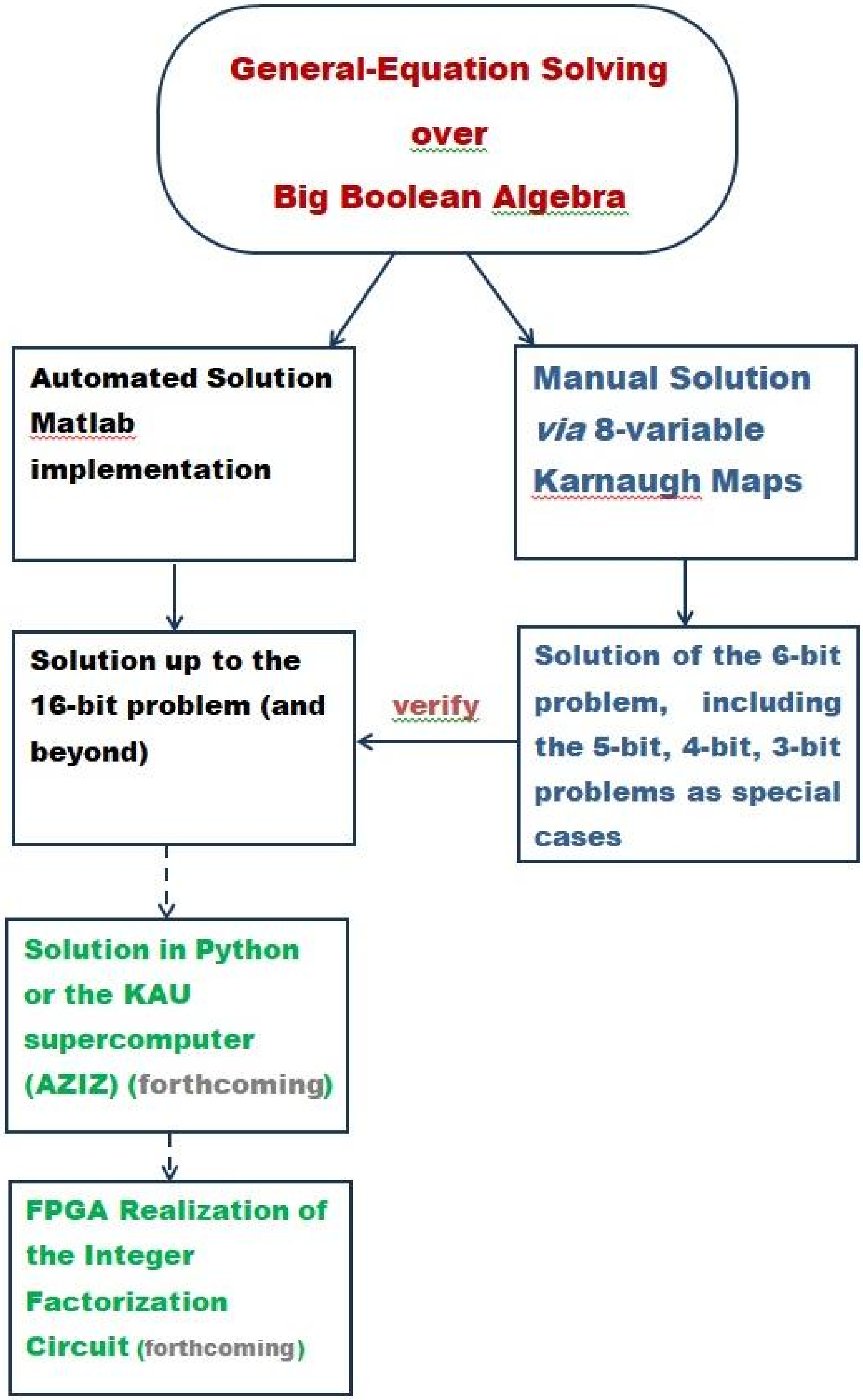


Fig. 5. An overview of our scheme for handling integer factorization

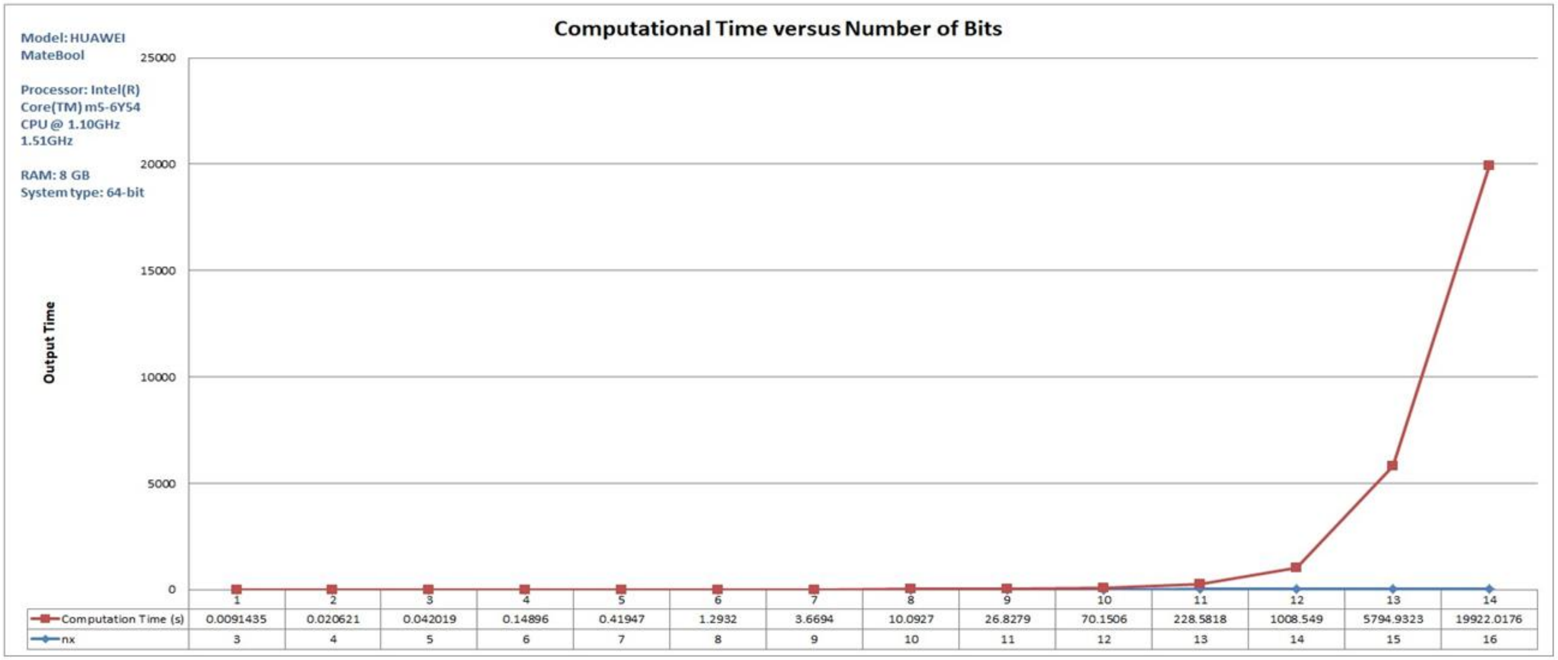


Fig. 6(a). Temporal complexity expressed as computational time versus the number of input bits

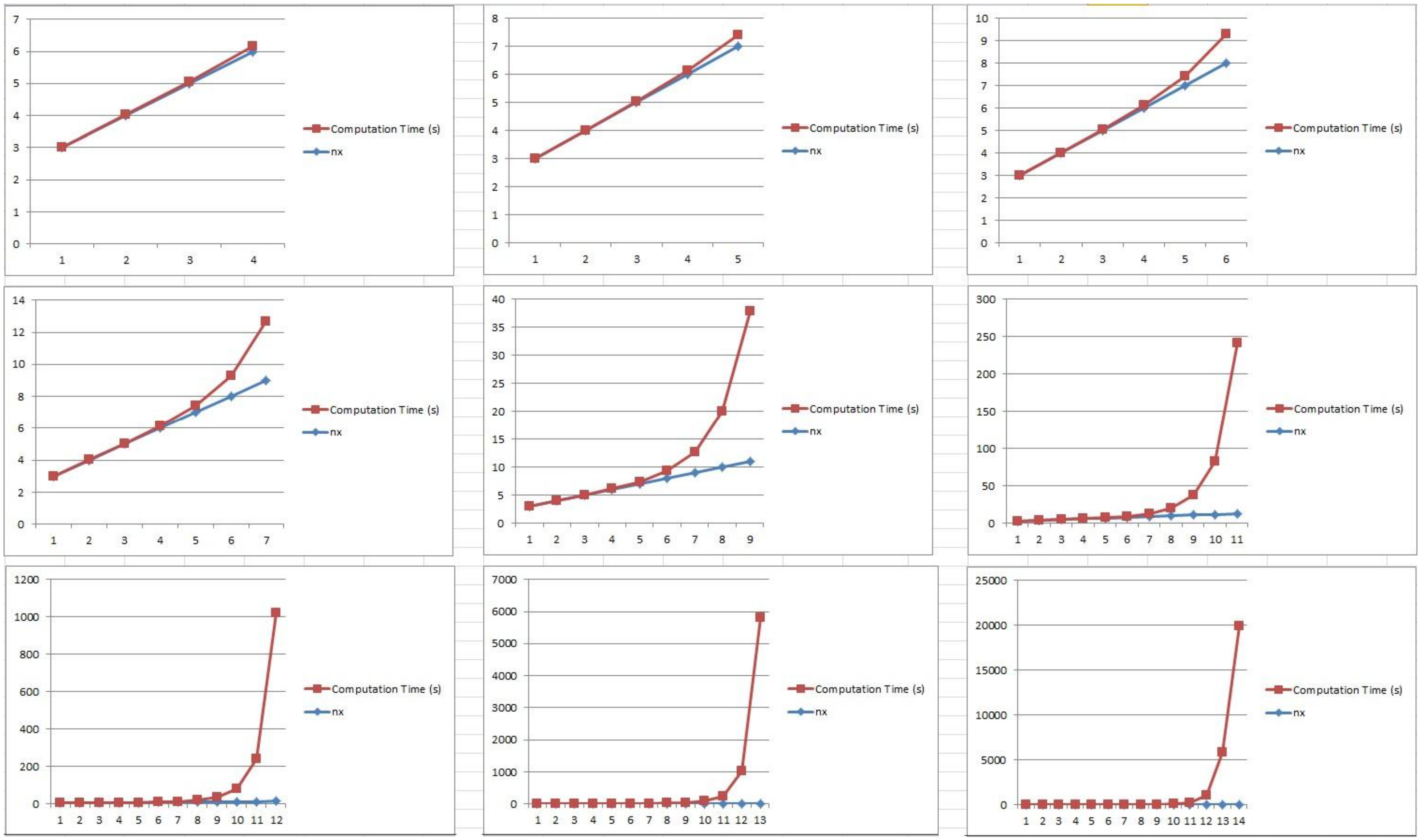


Fig. 6(b). Snapshots of the evolution of the temporal complexity in Fig. 6(a) as the number of input bits increases

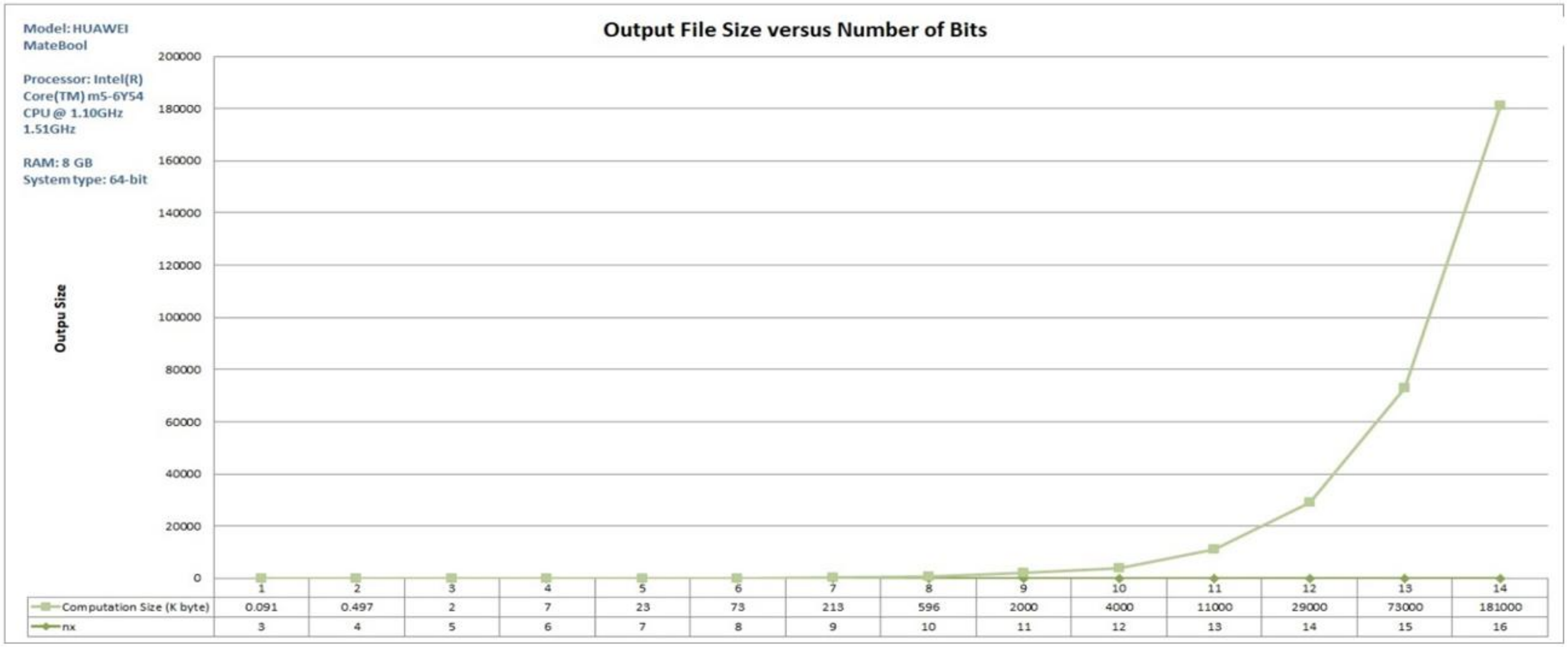


Fig. 7(a). Spatial complexity expressed as output file size versus the number of input bits

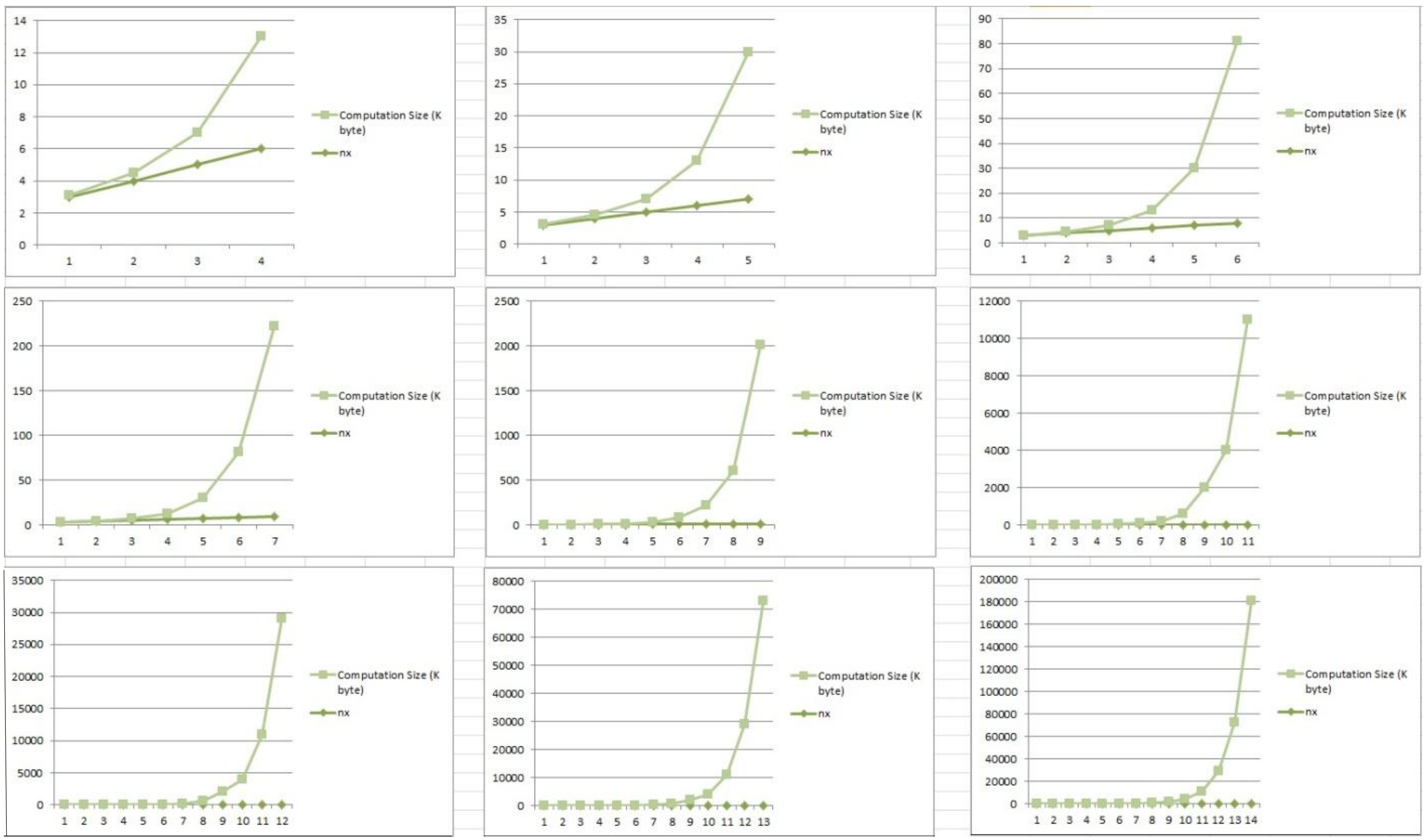


Fig. 7(b). Snapshots of the evolution of the spatial complexity in Fig. 7(a) as the number of input bits increases

We plan to try a new automated version using Python on AZIZ (King Abdulaziz University's super computer). The next step is to make an FPGA realization of the largest problem we manage to solve. So far, we were able to run the program for the 20-bit problem. The program might be used for larger problems as well, but both its execution time and output size are exponentially increasing, as clearly indicates by Figs. 6 and 7. The notorious "Curse of Dimensionality" is vividly demonstrated by the snapshots in Figs. 6(b) and 7(b) which show the evolution of the temporal and spatial complexities as the number of input bits increases.

5. Conclusions

Integer factorization is an intractable problem that might be handled in real time for small problems via hardware solution. Such a solution requires the extension of propositional logic to higher-order logics (e.g., first-order predicate logic) or the enlargement of two-valued Boolean algebra to a 'big' Boolean algebra. The paper derives a hardware circuit that factorizes a 6-bit integer **X** into two integers **Y** and **Z** of sizes 5 bits and 3 bits, respectively. The paper demonstrates that the resulting solution of the integer-factorization problem above includes the solutions of smaller problems as special cases. The paper builds on the experience gained in solving the 6-bit problem to design and implement a Matlab program to solve the general n-bit problem. The largest possible hardware circuit obtained via the automated solution is to be constructed, verified and tested. Such a hardware implementation (e.g., FPGA implementation) serves as a ready real-time look-up solution not only of the pertinent problem but also of all smaller problems.

Our contribution in this paper is admittedly a modest one and pertains mainly to a formulation of the integer factorization problem as a problem of Boolean equation-solving. This formulation gives a better insight into the problem and may provide opportunities to simplify the solution in the future. Our current solution method employs a strategy implicitly equivalent to that of a look-up table (albeit, with a more efficient enumeration). That is a main reason for the quick growth in the complexity of the solution.

The complexity in our solution comes from two sources. One involves the task of finding the Boolean expressions for the solution. This task is slow, but it has to be done once, and only once, for a problem of a given size. The other source of complexity is the size of the obtained expressions for the solutions. This is repeatedly unavoidable and would make a hardware implementation impractical for large problems due to large memory and time requirements. A possible way to reduce the complexity of our solution is to find the smallest factor (greater than 1) of the integer to be factorized and then, recursively, find the next factor, and so on.

Competing Interests

Authors have declared that no competing interests exist.

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APPENDIX A

Integer-Factorization Matlab Code

test01.m

```
close all;clear;clc;

nx=6;
%find n such that nn=2n or nn=2n-1

n=ceil(nx/2);

nz=n;
ny=nx-1;
c=zeros(1,2^nx);

d=[];
db=[];

for z=2:2^nz-1
        for y=z:2^ny-1
                if y*z<=2^nx-1
                        d=[d;y z y*z 0];
                        c(y*z+1)=c(y*z+1)+1;
                end
        end
end

ps=getparams(c);
k=find(c>=2);

for i=1:length(k)
        kk=find(d(:,3)==k(i)-1);
        d(kk,4)=(1:c(k(i)))';
end

ybits=mydec2bin(d(:,1),ny);
zbits=mydec2bin(d(:,2),nz);

for i=1:ny
        ex=['y' num2str(ny-i) '='];
        k=find(ybits(:,i)==1);
        dd=d(k,:);
        for j=1:size(dd,1)
        txt='';
        xbits=mydec2bin(dd(j,3),nx);
                for q=1:nx
                txt=[txt 'x' num2str(nx-q)];
                        if xbits(q)==0
                                txt=[txt ''''];
```

```
                end
            end
            if dd(j,4)>0
                atom=dd(j,3);
                param=ps{atom+1}{dd(j,4)};
                txt=[txt ' ' param];
            end
        if j>1, txt = [ ' + ' txt]; end
            ex=[ex txt];
        end
disp(ex);
end
```

getparams.m

```
function ps=getparams(c)

cc=c(c>=2);

nparams=sum(ceil(log2(cc)));

k=1;
for i=1:length(c)
    if c(i)<=1
        ps{i}='';
    else
        np=ceil(log2(c(i)));
        nmult=c(i);
        ps{i}=spantree(np,nmult,k);
        k=k+np;
    end
end
```

mydec2bin.m

```
function y=mydec2bin(x,b)

y=dec2bin(x,b);
y=double(y)==49;
y=double(y);
```

spantree.m

```
function pp=spantree(np,nmult,vindex)

%np=3;
%nmult=5;
%vindex=27;

tree=[-1 0 0 0 0];

node=1;
while true
```

```
          nnodes=size(tree,1);
          nendnodes=sum(tree(:,4));

          if nendnodes>=nmult, break; end
          if tree(node,2)==0 && tree(node,5)<np
                    tree(node,4)=0;
                    nnodes=nnodes+1;
                    newnode=[node 0 0 1 tree(node,5)+1];
                    tree(node,2)=nnodes;
                    tree=[tree;newnode];
                    node=nnodes;
          elseif tree(node,3)==0 && tree(node,5)<np
                    nnodes=nnodes+1;
                    newnode=[node 0 0 1 tree(node,5)+1];
                    tree(node,3)=nnodes;
                    tree=[tree;newnode];
                    nendnodes=nendnodes+1;
                    node=nnodes;
                    if nendnodes>=nmult, break; end
          else
                    node=tree(node,1);
          end
          if size(tree,1)>8, break; end
end

k=find(tree(:,4)==1);
for i=1:length(k)
          node=k(i);
          pp{i}='';
          while true
                    depth=tree(node,5);
                    px=['p' num2str(vindex+depth-1)];
                    pre=tree(node,1);
                    if tree(pre,3)==node
                              px=[px ''''];
                    end
                    pp{i}=[px pp{i}];
                    node=pre;
                    if pre==1, break; end
          end
end
```

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