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A New Class of Odd-Point Ternary Non-Stationary Schemes

Ghulam Mustafa[1*] and Mehwish Bari[1]

[1]Department of Mathematics, The Islamia University of Bahawalpur, Pakistan.

Research Article

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Abstract

In this paper we generate a new family of odd point ternary non-stationary interpolating subdivision schemes by using Lagrange identities. It is to be observed that the limiting ellipse, generated by proposed schemes compared to the existing non-stationary interpolating schemes, has less deviation from being an exact ellipse. The proposed non-stationary schemes are asymptotically equivalent to converging stationary schemes [1,2,3,4,5,6]. The performance and comparison of the schemes are verified by examples.

Keywords: Subdivision, interpolation, non-stationary, smoothness, tension control, conics reproduction

1 Introduction

Subdivision defines a curve or surface from an initial control mesh by recursive refinement. Starting with the coarse control points $\boldsymbol{f}^0 = \{f_i^0 \mid i \in \mathbb{Z}\}$, recursive application of the subdivision rule S_k defines a new set of points $\boldsymbol{f}^k = \left\{f_i^k \mid i \in \mathbb{Z}\right\}$, which can be written as

$$\boldsymbol{f}^k = S_k . . . S_0 \boldsymbol{f}^0, \quad k \in \mathbb{Z}_+.$$

A subdivision scheme is said to be stationary if S_k is the same regardless of k; otherwise it is called non-stationary [7]. The important schemes for applications should allow controlling the shape of the limit curve and being capable of reproducing families of curves widely used in Computer Graphics, such as conic sections and polynomials. Effectiveness of subdivision algorithms, their flexibility and ease make them appropriate for many relative computer graphics applications.

Here we present brief survey of existing literature. Jena et al. [7] proposed a 4-point binary non-stationary interpolating scheme. This scheme reproduces elements of the linear space spanned by $\{1, sin(\alpha x), \cos(\alpha x)\}$. Beccari et al. [8] proposed a non-stationary uniform tension controlled

**Corresponding author: ghulam.mustafa@iub.edu.pk, mehwishbari@yahoo.com;*

interpolating 4-point scheme with a single tension parameter having C^1 continuity. Daniel and Shunmugaraj [9] presented 3-point C^1 stationary and non-stationary schemes. They also proposed C^2 non-stationary approximating scheme in [10] and 4-point ternary interpolating non-stationary scheme [11] spanned by $\{1, sin(\alpha x), \cos(\alpha x)\}$. Beccari et al. [12] proposed 4-point ternary non-stationary interpolating scheme with tension control. They proposed the shape controlled ternary interpolatory subdivision in [13]. Further they presented the interpolating and approximating univariate subdivision by a unified framework [14]. Conti et al. [15] proposed approximating to interpolatory non-stationary subdivision schemes with the same generation properties. In [16], they also design interpolatory subdivision schemes by solving Bezout-like polynomial equations. Conti and Romani [17] presented algebraic conditions on non-stationary subdivision symbols for exponential polynomial reproduction. Charina et al. [18] proposed reproduction of exponential polynomials by multivariate non-stationary subdivision schemes with a general dilation matrix. Siddiqi and Younis [19,20] proposed 3-point ternary non-stationary approximating subdivision scheme and 4-point quaternary interpolating non-stationary subdivision scheme, respectively. Garnier [21] determined the characteristic elements for proper conic defined by three weighted points. He also proposed algorithms to represent an arc of central conic building. Garnier et al. [22] proposed iterative subdivisions for the construction of arcs, ellipses and hyperbolas in the Euclidean affine plane and mass points. Druoton et al. [23] constructed Dupin cyclide characteristic circles using non-stationary Iterated Function Systems. Furthermore, there are some other well known schemes [11,24,25,26] in the literature suitable for generating basic shapes in computer graphics.

Note that the existing non-stationary ternary interpolating schemes are the counterpart of existing stationary schemes (without parameter) but our proposed schemes are non-stationary counterpart of existing parametric stationary schemes. This property gives advantage to control the shape of the limit curve than the schemes without parameter.

We compare the existing non-stationary ternary schemes, suitable to generate ellipse, with the proposed schemes by the following strategy:

- Introduce deviation error function
- Refine initial control polygon with 4, 5 and 6 initial control points by different non-stationary schemes to generate ellipse
- Compute deviation error and compare the results.

We conclude that the limiting ellipse to be an exact ellipse, generated by the proposed scheme, has less deviation error as compared with the limiting ellipse produced by the different non-stationary ternary schemes.

This paper is organized as follows. In Section 2, we construct some results which are useful for Section 3. In Section 3, we present odd point non-stationary ternary interpolating schemes providing the user with a tension parameter that, when increased within its range of definition, can generate continuous limit curves showing considerable variations. Section 3 also provides the convergence of proposed interpolating schemes. Section 4 is devoted for comparison and conclusion.

2 Preliminaries

Given a set of control points $\boldsymbol{f}^0 = \{f_i^0 \in \mathbb{R} : i \in \mathbb{Z}\}$ at level 0, a ternary subdivision scheme for designing curves generates recursively a new set of control points $\boldsymbol{f}^{k+1} = \{f_{i+1}^k : i \in \mathbb{Z}\}$ at the $(k+1)^{th}$ level by a subdivision rule:

$$f_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-3j}^k f_i^k, \quad i \in \mathbb{Z}.$$

The set of coefficients $a^k = \{a_i^k : i \in \mathbb{Z}\}$ determines the subdivision rule at level k and is termed as the mask at k-th level. If the mask a^k is independent of k, the subdivision scheme S_{a^k} corresponding to the mask a^k is called stationary otherwise it is called non-stationary.

Theorem 2.1. [Equation (2.1), [24]] *Two ternary schemes S_{a^k} and S_{b^k} are asymptotically equivalent if*

$$\sum_{k=1}^{\infty} \left\| S_{a^k} - S_{b^k} \right\|_{\infty} < \infty,$$

where

$$\left\| S_{a^k} \right\|_{\infty} = max \left\{ \sum_{i\in\mathbb{Z}} |a_{3i}^k|, \quad \sum_{i\in\mathbb{Z}} |a_{3i+1}^k|, \ \sum_{i\in\mathbb{Z}} |a_{3i+2}^k| \right\}.$$

The proof of the following theorem is exactly similar by [11] to the proof given in (Theorem 8, [24]).

Theorem 2.2. *Let S_{a^k} and S_a be two asymptotically equivalent subdivision schemes having finite masks of the same support. Suppose S_{a^k} is non-stationary and S_a is stationary. If S_a is C^m and*

$$\sum_{k=0}^{\infty} 3^{mk} \left\| S_{a^k} - S_a \right\|_{\infty} < \infty,$$

then the non-stationary scheme S_{a^k} is C^m.

Deslauriers and Dubuc [1] presented the idea to construct subdivision schemes using Lagrange interpolation. Here we also use Lagrange polynomial to construct non-stationary schemes. First we define Lagrange fundamental polynomials of degree $2n-2$ and $2n-3$, for any integer $n \geq 1$, corresponding to nodes $\{x_j\}_{-(n-1)}^{n-1}$ and $\{x_j\}_{-(n-1)}^{n-2}$ respectively,

$$L_{x_m}^{2n-2}(x) = \prod_{x_j=-(n-1), x_m \neq x_j}^{n-1} \frac{x-x_j}{x_m-x_j}, \quad x_m = -(n-1), -(n-2), \dots, (n-1), \tag{2.1}$$

and

$$L_{x_m}^{2n-3}(x) = \prod_{x_j=-(n-2), x_m \neq x_j}^{n-1} \frac{x-x_j}{x_m-x_j}, \quad x_m = -(n-2), -(n-3), \dots, (n-1). \tag{2.2}$$

By using algebraic operations, we get following expressions:

$$\beta_1 = L_{x_m}^{2n-2}\left(\frac{1}{3}\right) = \frac{\frac{(-1)^{n-1}(3n-2)!}{3^{3n-3}(1-3x_m)(n-1)!}}{(-1)^{n-x_m-1}(n-x_m-1)!(n+x_m-1)!}, \tag{2.3}$$

$$\beta_2 = L_{x_m}^{2n-3}\left(\frac{1}{3}\right) = \frac{\frac{(-1)^{n-1}(3n-4)!}{3^{3n-5}(1-3x_m)(n-2)!}}{(-1)^{n-x_m-1}(n-x_m-1)!(n+x_m-2)!} = \frac{\tau_1}{\tau_2}. \tag{2.4}$$

Further, for $x_m = -n+1$ in (2.3), we get

$$\beta_3 = L_{-n+1}^{2n-2}\left(\frac{1}{3}\right) = \frac{(-1)^{-n+1}(3n-3)!}{3^{3n-3}(n-1)!(2n-2)!}. \tag{2.5}$$

Furthermore, we have

$$\tau_3 = \beta_1 - \beta_2 = \frac{(-1)^{x_m}(3n-3)!}{3^{3n-3}(n-1)!(n-x_m-1)!(n+x_m-1)!}. \tag{2.6}$$

$$A_{x_m} = \frac{\beta_1-\beta_2}{\beta_3} = \frac{\tau_3}{\beta_3}. \tag{2.7}$$

For more detail, we may refer to [6]. By perturbing expression (2.4) and (2.7) with sine function we get

$$\tilde{A}_{n,x_m} = \frac{\sin\left(\frac{\tau_1}{3^{k+1}}\right)}{\sin\left(\frac{\tau_2}{3^{k+1}}\right)}, \tag{2.8}$$

$$\hat{A}_{n,x_m} = \frac{\sin\left(\frac{\tau_3}{3^{k+1}}\right)}{\sin\left(\frac{\beta_3}{3^{k+1}}\right)}, \tag{2.9}$$

where $x_m = -n+2, \dots, n-1$ and $n \geq 1$ is any integer.

3 (2n-1)-Point Ternary Interpolating Scheme

In this section, we present general explicit formulae to construct the mask of $(2n-1)$ −point ternary interpolating subdivision scheme.

Given $n \geq 2$, the mask of following $(2n-1)$ −point ternary interpolating scheme

$$\begin{cases} f_{3i-1}^{k+1} = \sum_{x_m=-(n-1)}^{n-1} \eta_{-x_m}^{k,2n-1} f_{i+x_m}^{k}, \\ f_{3i}^{k} = f_i^{k}, \\ f_{3i+1}^{k+1} = \sum_{x_m=-(n-1)}^{n-1} \eta_{x_m}^{k,2n-1} f_{i+x_m}^{k}, \end{cases} \tag{3.1}$$

can be generated by

$$\begin{cases} \eta_{-n+1}^{k,2n-1} = \dfrac{\sin\left(\frac{\omega}{3^{k+1}}\right)}{\sin\left(\frac{1}{3^{k+1}}\right)}, \qquad \omega < 1, \\ \eta_{x_m}^{k,2n-1} = \tilde{A}_{n,x_m} + \hat{A}_{n,x_m}\omega, \qquad x_m = -n+2, \ldots, n-1, \end{cases} \tag{3.2}$$

where $\tilde{A}_{n,x_m}$ and $\hat{A}_{n,x_m}$ are defined by (2.8) and (2.9).

Examples:

Substituting $n = 2$ in (3.1) and (3.2), we get new 3-point ternary interpolating scheme with free parameter ω

$$\begin{aligned} f_{3i-1}^{k+1} &= \eta_1^{k,3} f_{i-1}^k + \eta_0^{k,3} f_i^k + \eta_{-1}^{k,3} f_{i+1}^k, \\ f_{3i}^{k+1} &= f_i^k, \\ f_{3i+1}^{k+1} &= \eta_{-1}^{k,3} f_{i-1}^k + \eta_0^{k,3} f_i^k + \eta_1^{k,3} f_{i+1}^k, \end{aligned} \tag{3.3}$$

where

$$\eta_{-1}^{k,3} = \frac{\sin\left(\frac{\omega}{3^{k+1}}\right)}{\sin\left(\frac{1}{3^{k+1}}\right)},$$

$$\eta_0^{k,3} = \frac{\sin\left(\frac{2}{3.3^{k+1}}\right)}{\sin\left(\frac{1}{3^{k+1}}\right)} - \frac{\sin\left(\frac{2}{9.3^{k+1}}\right)}{\sin\left(\frac{1}{9.3^{k+1}}\right)}\omega$$

and

$$\eta_1^{k,3} = \frac{\sin\left(\frac{1}{3.3^{k+1}}\right)}{\sin\left(\frac{1}{3^{k+1}}\right)} + \omega.$$

Substituting $n = 3$ in (3.1) and (3.2), we get new 5-point ternary interpolating scheme with free parameter ω

$$\begin{aligned} f_{3i-1}^{k+1} &= \eta_2^{k,5} f_{i-2}^k + \eta_1^{k,5} f_{i-1}^k + \eta_0^{k,5} f_i^k + \eta_{-1}^{k,5} f_{i+1}^k + \eta_{-2}^{k,5} f_{i+2}^k, \\ f_{3i}^{k+1} &= f_i^k, \\ f_{3i+1}^{k+1} &= \eta_{-2}^{k,5} f_{i-2}^k + \eta_{-1}^{k,5} f_{i-1}^k + \eta_0^{k,5} f_i^k + \eta_1^{k,5} f_{i+1}^k + \eta_2^{k,5} f_{i+2}^k, \end{aligned} \tag{3.4}$$

where

$$\eta_{-2}^{k,5} = \frac{\sin\left(\frac{\omega}{3^{k+1}}\right)}{\sin\left(\frac{1}{3^{k+1}}\right)},$$
$$\eta_{-1}^{k,5} = \frac{\sin\left(\frac{1}{3^{k+1}}\left[-\frac{5}{81} - 4\omega\right]\right)}{\sin\left(\frac{1}{3^{k+1}}\right)},$$
$$\eta_{0}^{k,5} = \frac{\sin\left(\frac{1}{3^{k+1}}\left[\frac{20}{27} + 6\omega\right]\right)}{\sin\left(\frac{1}{3^{k+1}}\right)},$$
$$\eta_{1}^{k,5} = \frac{\sin\left(\frac{1}{3^{k+1}}\left[\frac{10}{27} - 4\omega\right]\right)}{\sin\left(\frac{1}{3^{k+1}}\right)}$$

and

$$\eta_{2}^{k,5} = \frac{\sin\left(\frac{1}{3^{k+1}}\left[-\frac{4}{81} + \omega\right]\right)}{\sin\left(\frac{1}{3^{k+1}}\right)}.$$

Remark 3.1. The sum of weights of the scheme (3.3) tends to one as $k \to \infty$.

$$\xi^{\mathrm{k}} = \eta_{-1}^{k,3} + \eta_{0}^{k,3} + \eta_{1}^{k,3} = \frac{\sin\left(\frac{\omega}{3^{k+1}}\right)}{\sin\left(\frac{1}{3^{k+1}}\right)} + \frac{\sin\left(\frac{2}{3.3^{k+1}}\right)}{\sin\left(\frac{1}{3^{k+1}}\right)} - \frac{\sin\left(\frac{2}{9.3^{k+1}}\right)}{\sin\left(\frac{1}{9.3^{k+1}}\right)}\omega + \frac{\sin\left(\frac{1}{3.3^{k+1}}\right)}{\sin\left(\frac{1}{3^{k+1}}\right)} + \omega.$$

This implies

$$\xi^{\mathrm{k}} = \frac{\sin\left(\frac{\omega}{3^{k+1}}\right)}{\sin\left(\frac{1}{3^{k+1}}\right)} + \frac{\sin\left(\frac{2}{3.3^{k+1}}\right)}{\sin\left(\frac{1}{3^{k+1}}\right)} - 2\omega\cos\left(\frac{1}{9.\,3^{k+1}}\right) + \frac{\sin\left(\frac{1}{3.3^{k+1}}\right)}{\sin\left(\frac{1}{3^{k+1}}\right)} + \omega.$$

Using the inequalities $\sin a \geq a\cos a$ for $0 \leq a \leq \frac{\pi}{2}$, $\frac{1}{\sin a} \geq \frac{1}{a}$ for $0 \leq a \leq \frac{\pi}{2}$, we have

$$\xi^{\mathrm{k}} \geq \frac{\frac{\omega}{3^{k+1}}\cos\left(\frac{\omega}{3^{k+1}}\right)}{\frac{1}{3^{k+1}}} + \frac{\frac{2}{3.3^{k+1}}\cos\left(\frac{2}{3.3^{k+1}}\right) +}{\frac{1}{3^{k+1}}} - 2\omega\cos\left(\frac{1}{9.\,3^{k+1}}\right) + \frac{\frac{1}{3.3^{k+1}}\cos\left(\frac{1}{3.3^{k+1}}\right) +}{\frac{1}{3^{k+1}}} + \omega.$$

This implies

$$\xi^{\mathrm{k}} \geq \omega\cos\left(\frac{\omega}{3^{k+1}}\right) + \left(\frac{2}{3}\right)\cos\left(\frac{2}{3.\,3^{k+1}}\right) - 2\omega\cos\left(\frac{1}{9.\,3^{k+1}}\right) + \left(\frac{1}{3}\right)\cos\left(\frac{1}{3.\,3^{k+1}}\right) + \omega.$$

Thus $\xi^{\mathrm{k}} \geq 1$ when $k \to \infty$.

Again, using inequalities $\csc a \leq \frac{1}{a\cos a}$ for $0 \leq a \leq \frac{\pi}{2}$, $\sin a \leq a$ for $0 \leq a \leq \frac{\pi}{2}$, we have

$$\xi^{\mathrm{k}} \leq \frac{\frac{\omega}{3^{k+1}}}{\frac{1}{3^{k+1}}\cos\left(\frac{1}{3^{k+1}}\right)} + \frac{\frac{2}{3.3^{k+1}}}{\frac{1}{3^{k+1}}\cos\left(\frac{1}{3^{k+1}}\right)} - 2\omega\cos\left(\frac{1}{9.3^{k+1}}\right) + \frac{\frac{1}{3.3^{k+1}}}{\frac{1}{3^{k+1}}\cos\left(\frac{1}{3^{k+1}}\right)} + \omega.$$

This implies

$$\xi^{\mathrm{k}} \leq \frac{\omega}{\cos\left(\frac{1}{3^{k+1}}\right)} + \frac{2}{3\cos\left(\frac{1}{3^{k+1}}\right)} - 2\omega\cos\left(\frac{1}{9.3^{k+1}}\right) + \frac{1}{3\cos\left(\frac{1}{3^{k+1}}\right)} + \omega.$$

Thus $\xi^{\mathrm{k}} \leq 1$ when $k \to \infty$. So $\xi^{\mathrm{k}} = 1$ for $k \to \infty$.

Similarly, the sum of weights of the scheme (3.4) tends to one as $k \to \infty$.

3.1 Convergence of 3- and 5-Point Ternary Schemes

Here first we prove some lemmas by using following inequalities:-

$\frac{\sin a}{\sin b} \geq \frac{a}{b}$ for $0 < a \leq b < \frac{\pi}{2}$, $a \csc a < b \csc b$ for $0 < a < b < \frac{\pi}{2}$ and $\cos a < \frac{\sin a}{a}$ (or $\csc a < \frac{1}{a\cos a}$) for $0 < a < \frac{\pi}{2}$. Then with the help of these lemmas we will show that proposed non-stationary schemes are asymptotically equivalent to existing stationary schemes.

Lemma 3.1.

For 3-point non-stationary scheme (3.3) following inequalities hold:

(i) $\omega \leq \eta_{-1}^{k,3} \leq \frac{\omega}{\cos\left(\frac{1}{3^{k+1}}\right)}$

(ii) $\frac{2}{3} - 2\omega \leq \eta_{0}^{k,3} \leq \frac{2}{3\cos\left(\frac{1}{3^{k+1}}\right)} - \frac{2\omega}{\cos\left(\frac{1}{9.3^{k+1}}\right)}$

(iii) $\frac{1}{3} + \omega \leq \eta_{1}^{k,3} \leq \frac{1}{3\cos\left(\frac{1}{3^{k+1}}\right)} + \omega.$

Proof. We present the proof of (i) and the proof of (ii) and (iii) are similar. Note that

$$\frac{\sin\left(\frac{\omega}{3^{k+1}}\right)}{\sin\left(\frac{1}{3^{k+1}}\right)} \geq \frac{\frac{\omega}{3^{k+1}}}{\frac{1}{3^{k+1}}} = \omega.$$

Again consider

$$\frac{\sin\left(\frac{\omega}{3^{k+1}}\right)}{\sin\left(\frac{1}{3^{k+1}}\right)} \leq \frac{\omega}{3^{k+1}}\csc\left(\frac{1}{3^{k+1}}\right) \leq \frac{\frac{\omega}{3^{k+1}}}{\frac{1}{3^{k+1}}\cos\left(\frac{1}{3^{k+1}}\right)} = \frac{\omega}{\cos\left(\frac{1}{3^{k+1}}\right)}.$$

This proves (i).

From Lemma 3.1, we get following lemma.

Lemma 3.2.

For scheme (3.3) with $\omega = -\frac{1}{3} + u$, for $u \in \left(\frac{2}{9}, \frac{1}{3}\right)$, we have

$(i)\ -\frac{1}{3} + u \leq \gamma_{-1}^{k,3} \leq \frac{-\frac{1}{3}+u}{\cos\left(\frac{1}{3^{k+1}}\right)}$

$(ii)\ \frac{4}{3} - 2u \leq \gamma_{0}^{k,3} \leq \frac{\frac{4}{3}-2u}{\cos\left(\frac{1}{3^{k+1}}\right)}$

$(iii)\ u \leq \gamma_{1}^{k,3} \leq \frac{u}{\cos\left(\frac{1}{3^{k+1}}\right)}$.

Lemma 3.3.

For scheme (3.3) following inequalities also hold:

$(i)\ \left|\eta_{-1}^{k,3} - \omega\right| \leq C_0 \left(\frac{1}{3^{2k}}\right)$

$(ii)\ \left|\eta_{0}^{k,3} - \left(\frac{2}{3} - 2\omega\right)\right| \leq C_1 \left(\frac{1}{3^{2k}}\right)$

$(iii)\ \left|\eta_{1}^{k,3} - \left(\frac{1}{3} + \omega\right)\right| \leq C_2 \left(\frac{1}{3^{2k}}\right)$

where constants C_0, C_1 and C_2 are independent of k.

Proof. The inequality (i) can be proved by using (i) of Lemma 3.1:

$$\left|\eta_{-1}^{k,3} - \omega\right| = \omega \left[\frac{1 - \cos\left(\frac{1}{3^{k+1}}\right)}{\cos\left(\frac{1}{3^{k+1}}\right)}\right] \leq \omega \left[\frac{2\sin^2\left(\frac{1}{2}\frac{1}{3^{k+1}}\right)}{\cos\left(\frac{1}{3^{k+1}}\right)}\right] \leq \left(\frac{1}{4}\right)\left(\frac{2\omega}{3^{2k+2}\cos\left(\frac{1}{3^{k+1}}\right)}\right).$$

This implies

$$\left|\eta_{-1}^{k,3} - \omega\right| \leq \left(\frac{1}{3^{2k}}\right)\left(\frac{\omega}{18\cos(1)}\right) \leq e_0 \left(\frac{1}{3^{2k}}\right).$$

The proofs of (ii) and (iii) are similar.

From Lemma 3.3, we get following lemma.

Lemma 3.4.

For scheme (3.3) with $\omega = -\frac{1}{3} + u$, *for* $u \in \left(\frac{2}{9}, \frac{1}{3}\right)$, *following inequalities hold:*

(i) $\left|\eta_{-1}^{k,3} - \left(-\frac{1}{3} + u\right)\right| \leq \acute{C}_0 \left(\frac{1}{3^{2k}}\right)$

(ii) $\left|\eta_{0}^{k,3} - \left(\frac{4}{3} - 2u\right)\right| \leq \acute{C}_1 \left(\frac{1}{3^{2k}}\right)$

(iii) $\left|\eta_{1}^{k,3} - u\right| \leq \acute{C}_2 \left(\frac{1}{3^{2k}}\right)$

where constants $\acute{C}_0$, $\acute{C}_1$ *and* $\acute{C}_2$ *are independent of* k.

Remark 3.2. From (i)-(iii) of Lemma 3.4, we observe that

$\eta_{-1}^{k,3} \to -\frac{1}{3} + u$, $\eta_{0}^{k,3} \to \frac{4}{3} - 2u$ and $\eta_{1}^{k,3} \to u$, as $k \to \infty$.

This means that the mask of the scheme (3.3) with $\omega = -\frac{1}{3} + u$, for $u \in \left(\frac{2}{9}, \frac{1}{3}\right)$ converge to the mask of the scheme [4].

Similarly, it is to be mentioned that for $n = 2$, $\omega = a$ where $a = b - \frac{1}{3}$, $\omega = -\frac{1}{9}$, $\omega = -\frac{1}{3} + u$ and $\omega = -\frac{1}{3} + u$ in (3.1), (3.2) and by proving/using similar inequalities like in Lemma 3.1 and Lemma 3.3, we get non-stationary counter part of stationary schemes of [2,3,4,6] respectively.

Theorem 3.5. *The proposed 3-point non-stationary scheme (3.3) with* $\omega = -\frac{1}{3} + u$, $u \in \left(\frac{2}{9}, \frac{1}{3}\right)$ *is* C^1.

Proof. We claim that

$$\sum_{k=0}^{\infty} 3^k \left\|S_{a^k} - S_a\right\|_{\infty} < \infty,$$

where

$$\left\|S_{a^k} - S_a\right\|_{\infty} = max \left\{ \sum_{j \in \mathbb{Z}} \left|a_{i-3j}^{k} - a_{i-3j}\right| : i \in 0, 1, 2 \right\}.$$

From scheme S_{a^k} defined by (3.3) with $\omega = -\frac{1}{3} + u$ and scheme S_a of [4]

$$\sum_{k=0}^{\infty} 3^k \left\|S_{a^k} - S_a\right\|_{\infty} = \sum_{k=0}^{\infty} 3^k \left\{ \left|\eta_{-1}^{k,3} - \left(-\frac{1}{3} + u\right)\right| + \left|\eta_{0}^{k,3} - \left(\frac{4}{3} - 2u\right)\right| + \left|\eta_{1}^{k,3} - u\right| \right\}.$$

From (i) of Lemma 3.4, it follows that

$$\sum_{k=0}^{\infty} 3^k \left|\eta_{-1}^{k,3} - \left(-\frac{1}{3}+u\right)\right| \leq \sum_{k=0}^{\infty} 3^k \acute{C}_0 \left(\frac{1}{3^{2k}}\right) < \infty.$$

Similarly from (ii) and (iii) of Lemma (3.4), we see that other terms are also less than ∞. Hence $\sum_{k=0}^{\infty} 3^k \left\|S_{a^k} - S_a\right\|_{\infty} < \infty$. Since stationary scheme of [4] is C^1 therefore by Theorem 2.2, proposed scheme (3.3) with $\omega = -\frac{1}{3} + u$ is C^1.

Now we will discuss the continuity of 5-point scheme (3.4). For this first we will prove the following lemmas. Proof of these lemmas are similar to the proof of Lemmas 3.1-3.4.

Lemma 3.6.

For 5-point non-stationary scheme (3.4), following inequalities hold:

(i) $\omega \leq \eta_{-2}^{k,5} \leq \frac{\omega}{\cos\left(\frac{1}{3^{k+1}}\right)}$

(ii) $-\frac{5}{81} - 4\omega \leq \eta_{-1}^{k,5} \leq -\frac{5}{81\cos\left(\frac{6}{3^{k+1}}\right)} - \frac{4\omega}{\cos\left(\frac{5}{243.3^{k+1}}\right)}$

(iii) $\frac{20}{27} + 6\omega \leq \eta_{0}^{k,5} \leq \frac{20}{27\cos\left(\frac{2}{3^{k+1}}\right)} + \frac{6\omega}{\cos\left(\frac{5}{243.3^{k+1}}\right)}$

(iv) $\frac{10}{27} - 4\omega \leq \eta_{1}^{k,5} \leq \frac{10}{27} - 4\omega\cos\left(\frac{20}{243.3^{k+1}}\right)$

(v) $-\frac{4}{81\cos\left(\frac{6}{3^{k+1}}\right)} + \omega \leq \eta_{2}^{k,5} \leq -\frac{4}{81} + \omega.$

From Lemma (3.6), we get following lemma:

Lemma 3.7.

For scheme (3.4) with $\omega = \frac{4}{81} + u$, for $u \in \left(-\frac{5}{108}, -\frac{7}{162}\right)$, we have

(i) $\frac{4}{81} + u \leq \eta_{-2}^{k,5} \leq \frac{\frac{4}{81}+u}{\cos\left(\frac{1}{3^{k+1}}\right)}$

(ii) $-\frac{7}{27} - 4u \leq \eta_{-1}^{k,5} \leq \frac{-\frac{7}{27}-4u}{\cos\left(\frac{1}{3^{k+1}}\right)}$

(iii) $\frac{28}{27} + 6u \leq \eta_{0}^{k,5} \leq \frac{\frac{28}{27}+6u}{\cos\left(\frac{1}{3^{k+1}}\right)}$

$(iv)\, \frac{14}{81} - 4u \leq \eta_1^{k,5} \leq \frac{\frac{14}{81} - 4u}{\cos\left(\frac{1}{3^{k+1}}\right)}$

$(v)\; u \leq \eta_2^{k,5} \leq \frac{u}{\cos\left(\frac{1}{3^{k+1}}\right)}.$

Lemma 3.8.

For scheme (3.4) following inequalities also hold:

$(i)\; \left|\eta_{-2}^{k,5} - \omega\right| \leq g_0\left(\frac{1}{3^{2k}}\right)$

$(ii)\; \left|\eta_{-1}^{k,5} - \left(-\frac{5}{81} - 4\omega\right)\right| \leq g_1\left(\frac{1}{3^{2k}}\right)$

$(iii)\; \left|\eta_0^{k,5} - \left(\frac{20}{27} + 6\omega\right)\right| \leq g_2\left(\frac{1}{3^{2k}}\right)$

$(iv)\; \left|\eta_1^{k,5} - \left(\frac{10}{27} - 4\omega\right)\right| \leq g_3\left(\frac{1}{3^{2k}}\right)$

$(v)\; \left|\eta_2^{k,5} - \left(-\frac{4}{81} + \omega\right)\right| \leq g_4\left(\frac{1}{3^{2k}}\right)$

Where constants g_0, g_1, g_2, g_3 *and* g_4 *are independent of* k.

From Lemma 3.8, we get following lemma.

Lemma 3.9.

For scheme (3.4) with $\omega = \frac{4}{81} + u$*, for* $u \in \left(-\frac{5}{108}, -\frac{7}{162}\right)$*, following inequalities hold:*

$(i)\; \left|\eta_{-2}^{k,5} - \left(\frac{4}{81} + u\right)\right| \leq \acute{g}_0\left(\frac{1}{3^{2k}}\right)$

$(ii)\; \left|\eta_{-1}^{k,5} - \left(-\frac{7}{27} - 4u\right)\right| \leq \acute{g}_1\left(\frac{1}{3^{2k}}\right)$

$(iii)\; \left|\eta_0^{k,5} - \left(\frac{28}{27} + 6u\right)\right| \leq \acute{g}_2\left(\frac{1}{3^{2k}}\right)$

$(iv)\; \left|\eta_1^{k,5} - \left(\frac{14}{81} - 4u\right)\right| \leq \acute{g}_3\left(\frac{1}{3^{2k}}\right)$

$(v)\; \left|\eta_2^{k,5} - u\right| \leq \acute{g}_4\left(\frac{1}{3^{2k}}\right)$

Where constants $\acute{g}_0$, $\acute{g}_1$, $\acute{g}_2$, $\acute{g}_3$ *and* $\acute{g}_4$ *are independent of* k.

***Remark* 3.3.** It is to be noted that for negative values of ω, the trigonometric inequalities does not affect the proof of our main Lemma 3.3 and Lemma 3.8.

***Remark* 3.4.** From $(i)-(v)$ of Lemma 3.9, we observe that the mask of scheme (3.4) with $\omega = \frac{4}{81}+u, u \in \left(-\frac{5}{108}, -\frac{7}{162}\right)$ converge to the mask of the scheme S_a of [4].

$\eta_{-2}^{k,5} \to \frac{4}{81}+u, \eta_{-1}^{k,5} \to -\frac{7}{27}-4u, \eta_{0}^{k,5} \to \frac{28}{27}+6u, \eta_{1}^{k,5} \to \frac{14}{81}-4u$ and $\eta_{2}^{k,5} \to u,$ as $k \to \infty$.
Similarly, for $n=3$, $\omega = \frac{5}{243}$, $\omega = \frac{4}{81}+u$ and $\omega = \frac{4}{81}+u$ in (3.1), (3.2) and using similar inequalities as in Lemma 3.6 and Lemma 3.8, the proposed (3.4) scheme becomes non-stationary counterpart of the 5-point stationary schemes of [3,4,6] respectively.

Theorem 3.10. *The proposed 5-point non-stationary scheme (3.4) with* $\omega = \frac{4}{81}+u$, *for* $u \in \left(-\frac{5}{108}, -\frac{7}{162}\right)$ *is* C^2.

Proof. From scheme S_{a^k} defined by (3.4) with $\omega = \frac{4}{81}+u$, for $u \in \left(-\frac{5}{108}, -\frac{7}{162}\right)$ and scheme S_a of [4]

$$\sum_{k=0}^{\infty} 3^{2k} \left\|S_{a^k} - S_a\right\|_{\infty} = \sum_{k=0}^{\infty} 3^{2k} \left\{ \left|\eta_{-2}^{k,5} - \left(\frac{4}{81}+u\right)\right| + \left|\eta_{-1}^{k,5} - \left(-\frac{7}{27}-4u\right)\right| + \left|\eta_{0}^{k,5} - \left(\frac{28}{27}+6u\right)\right| + \left|\eta_{1}^{k,5} - \left(\frac{14}{81}-4u\right)\right| + \left|\eta_{2}^{k,5} - u\right| \right\}.$$

By using inequalities $(i)-(v)$ of Lemma 3.9, we see that $\sum_{k=0}^{\infty} 3^{2k} \left\|S_{a^k} - S_a\right\|_{\infty} < \infty$. Since stationary scheme of [4] is C^2 therefore by Theorem 2.2, proposed scheme (3.4) is C^2.

***Remark* 3.5. Relation with DD schemes:** The proposed non stationary ternary interpolating schemes are counter part of stationary DD schemes [1].

- From Lemma 3.1 and Lemma 3.3 with $\omega = 0$ and $k \to \infty$, we get 2-point ternary DD interpolating scheme.
- For $\omega = 0$ and $k \to \infty$ in Lemma 3.6 and Lemma 3.8, we get 4-point ternary DD interpolating scheme. Similarly, we get other DD ternary schemes.

4 Comparison and Conclusion

First we generate conic sections by using proposed schemes then we present comparison among proposed and existing non-stationary interpolating schemes. If the initial control points are chosen as values at equidistant points of a function $f(x) \in span\{\cos(\beta x),\ \sin(\beta x)\}, 0 < \beta < \pi$, then the limit function of the scheme is the original function. In particular, if the initial control points are equidistance points and lie on a circle or an ellipse, the scheme generates a circle or an ellipse respectively.

Numerical Examples: Here we take the set of equidistant points $f_i^0 = \left(a\cos\left(\frac{2i\pi}{N}\right), b\sin\left(\frac{2i\pi}{N}\right)\right)$, $i = 0,1,2,\dots,N, N \geq 4,$ as control points of initial control polygons of proposed 3-, 5-point non-stationary schemes. For $a = 2, \mathrm{b} = 1$, limiting curves generated by 3-, 5-point schemes are ellipses shown in Fig. 1(a) and 1(b). It is observed that for large values of ω limit curves generated by proposed 3-point scheme passes near the initial control polygon whereas for small values of ω limit curves generated by proposed 5-point scheme passes near the initial control polygon as shown in Fig. 1(a) and 1(b).

Analogously, by choosing a set of equidistant points from the parabolic equation $(y^2 = 4ax)$ for $\mathrm{a} = 1$ and hyperbolic equation $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right)$ where $\mathrm{a} = 3, \mathrm{b} = 2$ then limit curve is parabola and hyperbola as shown in Fig. 2(a) and 2(b).

Comparison: In following, we numerically compare the exactness of limiting ellipses generated by different non-stationary subdivision schemes by using following function.

$$D_k = \left|max\{f_i^k\} - min\{f_i^k\}\right|, \tag{4.1}$$

Where

$$f_i^k = \sqrt{(x_i^k + c)^2 + (y_i^k)^2} + \sqrt{(x_i^k - c)^2 + (y_i^k)^2}, \quad \text{for} \quad b^2 \equiv a^2 - c^2 \tag{4.2}$$

where $a = 2, b = 1$ are semi-major and semi-minor axis respectively and f_i^k are control points generated by subdivision scheme at k-th level of iteration for $k \geq 0$. If the initial control points f_i^0 lie on the ellipse then of course D_0 will be zero. If $D_k = 0$ for sufficiently large k then its mean scheme produce exact ellipse. If $D_k \neq 0$ then f_i^k do not lie on same ellipse. Since D_k measures the maximum deviation of limiting ellipse from being an exact ellipse therefore we can present comparison among different limiting ellipses generated by proposed and existing non-stationary schemes.

By taking four, five and six initial control points, we first generate limiting ellipses by proposed 3-, 5-point and existing non-stationary schemes of [7,8,11,12,13,26] then we compute deviation error D_k. Deviation of proposed 3-, 5-point ternary non-stationary interpolating schemes is calculated at parametric values -0.1122, 0.0228, respectively.

- In case of 4 initial control points: Limiting ellipses produced by different non-stationary are shown in Fig. 3(a-e) while deviation errors in graphical form are shown in Fig. 5(a) and Fig. 5(b).
- In case of 5 initial control points: Limiting ellipses are depicted in Fig. 3(f-j) while graphical representations of deviation errors are shown in Fig. 5(c) and 5(d).
- In case of 6 initial control points: Limiting ellipses are painted in Fig. 4(a-e) while graphical representation of deviation errors are shown in Fig. 5(e) and 5(f).

Deviation error in limiting ellipses, produced by different schemes with 4, 5 and 6 initial control points, in tabular form are shown in Table 1. From these figures and table it is clear that the limiting ellipses to be an exact ellipses, generated by proposed scheme, have less deviation compare to the limiting ellipses produced by the schemes of [7,8,11,12,13,26].

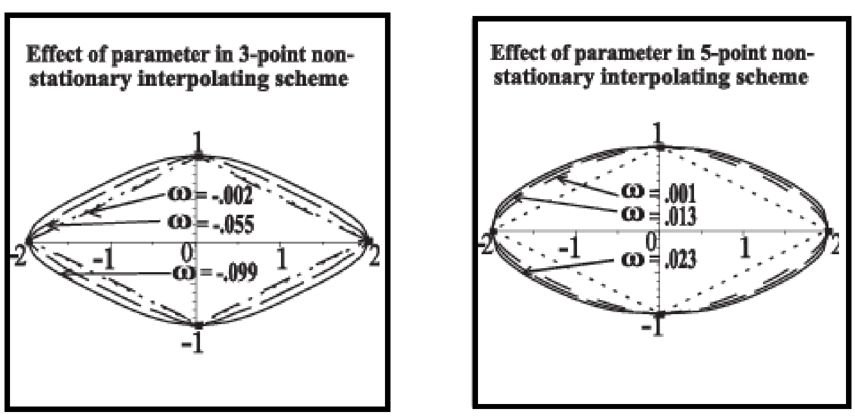
Table 1. Comparison of deviation error (D. Error) with existing non-stationary interpolating schemes: Here N represents the number of initial points of control polygon

Schemes	N	D. Error	N	D. Error	N	D. Error
4-point binary [7,8,26]	4	0.18501	5	0.20398	6	0.05910
3-point ternary [26]	4	0.30308	5	0.44910	6	0.17196
4-point ternary [11]	4	0.20160	5	0.20399	6	0.06478
4-point ternary [12]	4	0.19510	5	0.20365	6	0.06219
4-point ternary [13]	4	0.20165	5	0.20409	6	0.06483
3-point proposed	4	0.18957	5	0.19579	6	0.05865
5-point proposed	4	0.08104	5	0.04937	6	0.01106

4.1 Conclusion and Future Work

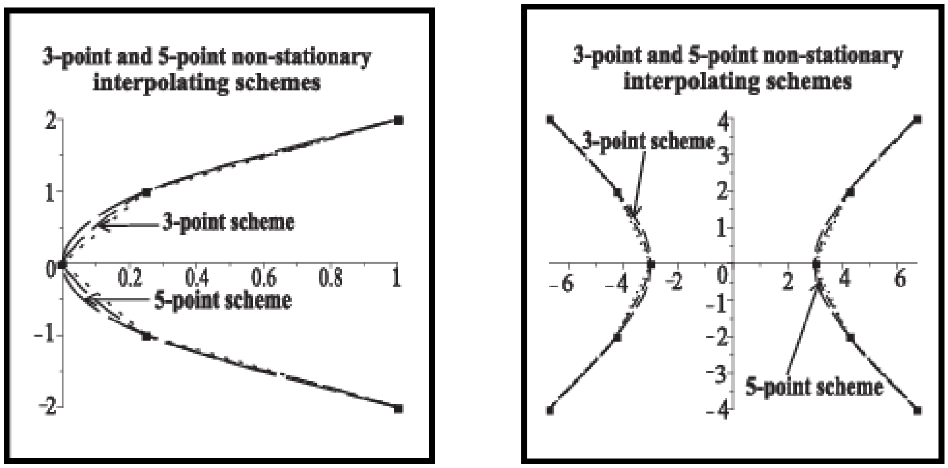
By using Lagrange identities we construct new families of univariate, ternary, non-stationary interpolating subdivision schemes for curve design with a single tension parameter which enable the scheme to produce more precise result. The proposed schemes are non-stationary counterpart of the stationary schemes [1,2,3,4,5,6] so the parametric ranges of continuity of proposed non-stationary schemes are same as of the counter stationary schemes. Fig. 1 illustrates that the proposed scheme gives great flexibility to geometric designers for the creation of smooth curves according to their own requirements by choosing appropriate value of parameter.

Here are some tips for future work proposed by the anonymous referee: Let γ_0 and γ_1 be two curves defined on [0,1]. Let $A = \gamma_0(1) = (-2, 2)$ and $B = \gamma_1(0) = (2,2)$, the components of tangent vector to γ_0(resp. γ_1) at A (resp. B) is (1, 1) (resp. (1, -1)). Is it easy to compute a subdivision of the conic which realizes a G^1 −blend between these two curves using proposed method in this article? Is it possible to choose a circular arc to blend these curves? If someone wants to subdivide a hyperbola arc, could someone have an end point on a branch and the other end point on the other branch? Articles [21,22,23] might help to find the answers of above questions.



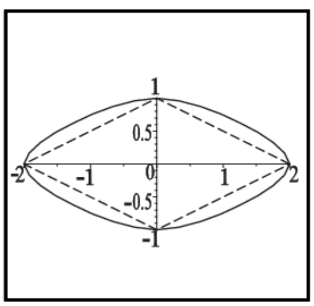
(a) Scheme (3.3) **(b) scheme (3.4)**

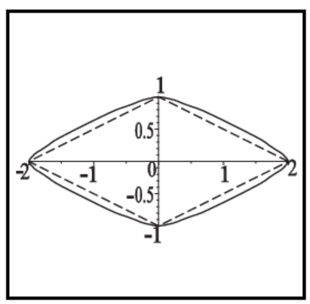
Fig. 1. (a) Shows the increase in tightness of the curve with increasing ω and (b) Shows the increase in tightness of the curve with decreasing ω.

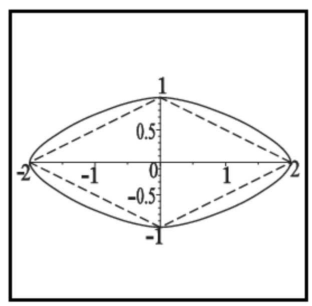


(a) (b)

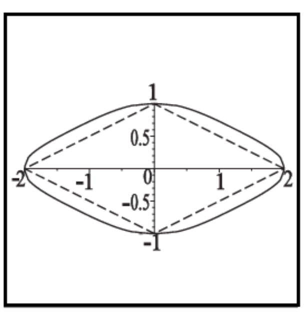
Fig. 2. Represents parabola and hyperbola by proposed 3-, 5-point non-stationary interpolating schemes after two iterations .

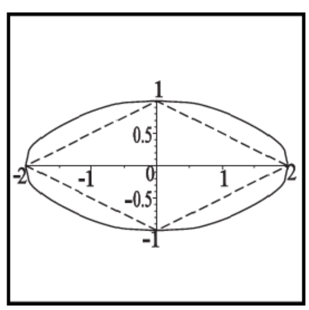




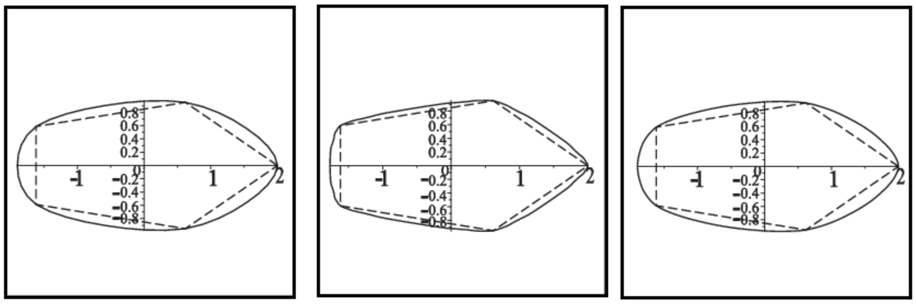


(a) [7,8,26] **(b) [26]** **(c) [11]**

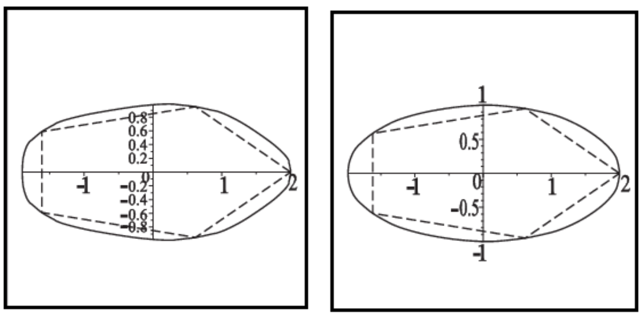




(d) Scheme (3.3) **(e) scheme (3.4)**

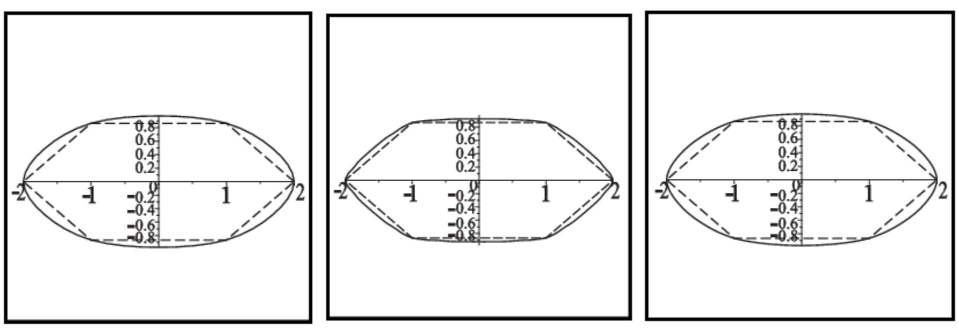


(f) [7,8,26] (g) [26] (h) [11]

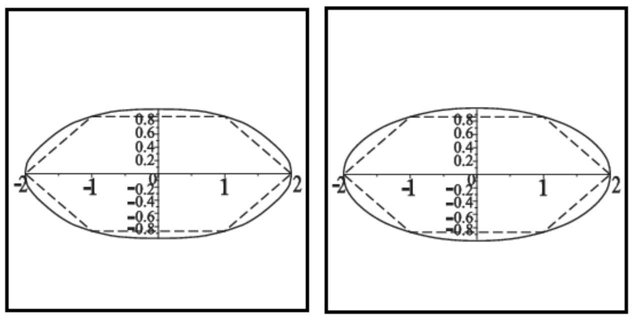


(i) Scheme (3.3) (j) scheme (3.4)

Fig. 3. (a) and (f) Shows the ellipse produced by non-stationary binary schemes of [7,8,26], (b), (g), (c) and (h) show the ellipse produced by non-stationary ternary schemes of [26,11], (d), (i), (e) and (j) show the ellipse produced by proposed 3-, 5-point non-stationary interpolating schemes respectively

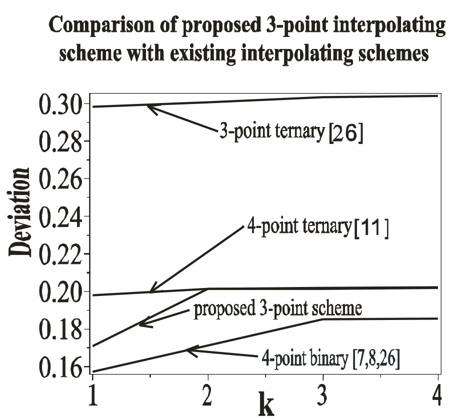


(a) [7,8,26] (b) [26] (c) [11]

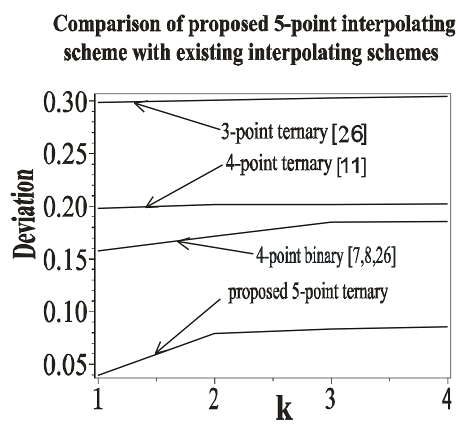


(d) Scheme (3.3) **(e) scheme (3.4)**

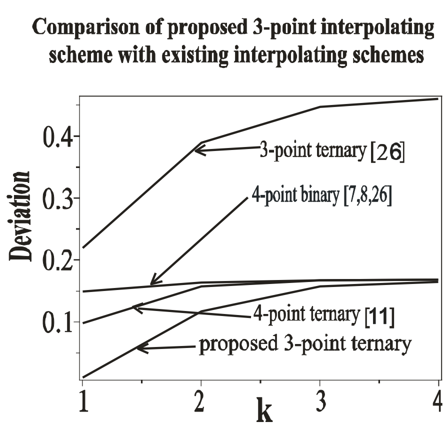
Fig. 4. (a) Shows the ellipse produced by non-stationary binary scheme of [7,8,26], (b) and (c) show the ellipse produced by non-stationary ternary scheme of [26,11], (d) and (e) show the ellipse produced by proposed 3-, 5-point non-stationary ternary interpolating schemes respectively



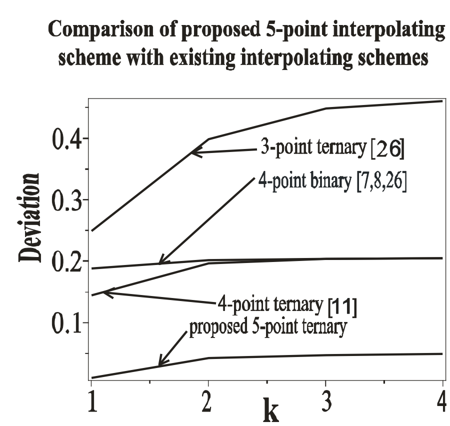
(a) 4 initial points



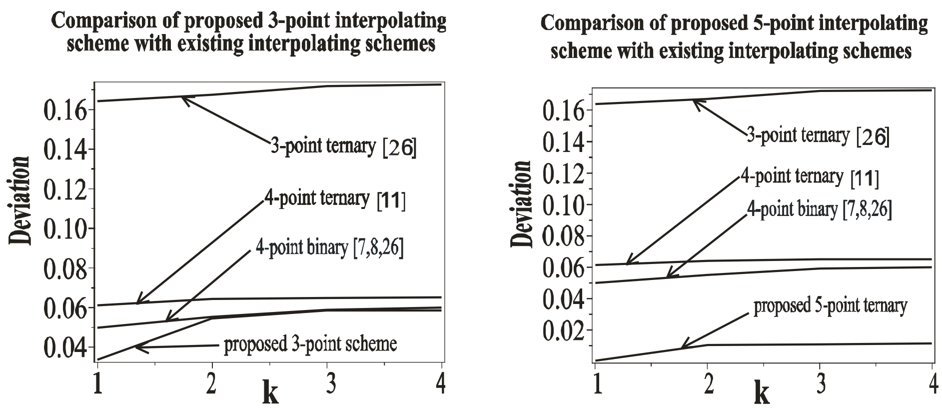
(b) 4 initial points



(c) 5 initial points



(d) 5 initial points



(e) 6 initial point **(f) 6 initial point**

Fig. 5. Graphs show the deviation of proposed and existing non-stationary interpolating schemes

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Competing Interests

Authors have declared that no competing interests exist.

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