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A Generalized Non-Stationary *4*-Point *b*-ary Approximating Scheme

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Research Article

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Abstract

A generalized non-stationary *4*-point *b*-ary approximating subdivision scheme is presented for even integer $b \geq 2$. Lagrange trigonometric polynomial plays a key role in computation of mask of the generalized scheme. The proposed schemes can be considered as non-stationary counterpart of existing stationary approximating schemes. Asymptotic equivalence technique is used for convergence analysis of the proposed schemes. Efficiency of proposed schemes is illustrated with the help of some examples.

Keywords: 4-point approximating, non-stationary, subdivision scheme, Lagrange, asymptotic equivalence.

1 Introduction

In the study of curve generating techniques subdivision is lime lighted due to its efficient and easy-to-use implementation. Subdivision schemes produce smooth curves by applying iterative refinements on set of control points. Subdivision schemes are classified into different categories upon their characteristics. For example if the points of limiting curve pass through initial control points then subdivision scheme is called interpolating otherwise approximating. Similarly, if the mask of the scheme does not vary with subdivision level then it is termed as stationary otherwise non-stationary.

There are many stationary subdivision schemes in literature but non-stationary schemes are gaining interest day by day in research community. Morin et al. [1] derived mask of binary approximating non-stationary subdivision scheme which unifies cubic splines, splines-in-tension and a certain class of trigonometric splines. Daniel and Shunmugaraj [2] offered 2-point binary and 3-point ternary non-stationary approximating schemes by using Lagrange polynomial. Daniel and Shunmugaraj [3] presented 2-point and 3-point binary non-stationary approximating subdivision scheme based on trigonometric B-spline basis function. Daniel and Shunmugaraj [4] also introduced 3-point binary non-stationary approximating scheme. We also refer to [5-9] for some other non-stationary scheme in the literature.

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Higher arity schemes are better than lower arity schemes because of their less computational cost and support as compared to lower arity schemes. Support of the scheme shows the region in which the limit curve/surface will change when a single control point is moved. The larger the support, the wider the influence of each control point. In general, schemes with compact support are preferred [10]. So the best way to get compact support is to raise arity. By taking this into account, we propose subdivision schemes of higher arity.

Mustafa and Rehman [11] presented general formulae for the mask of *(2b+4)*-point *n*-ary approximating and as well as interpolating subdivision schemes. These formulae provide mask of higher arity schemes and generalize lower arity schemes. Masks of stationary schemes have been derived based on Lagrange polynomial while we derive mask of 4-point even-ary non-stationary scheme using trigonometric Lagrange polynomial. Augsdörfer et al. [12] applied different variants on classical 4-point interpolating scheme [13] producing some subdivision scheme all of which are improvements on the original scheme. They have derived two non-stationary schemes one of them has continuity C^1 which is smaller than the continuity of proposed schemes and other has larger support than the support of proposed schemes. Pan et al. [14] presented a combined approximating and interpolating subdivision scheme. The connection between interpolating and approximating scheme is made by directly performing operations on geometric rules. This combined scheme is 4-point ternary stationary and has larger support than the proposed *4*-point quaternary non-stationary scheme.

In this paper, we present a generalized non-stationary *4*-point *b*-ary approximating subdivision scheme constructed by using trigonometric Lagrange polynomial. We show that proposed schemes are non-stationary counterpart of existing stationary schemes. We use theory of asymptotic equivalence to investigate the convergence of the schemes. Moreover, we also discuss some important properties of schemes like affine invariance, support and symmetry of basic limit function.

The paper is structured as follows. In Section 2, we listed some basic preliminaries. In Section 3, we introduce the generalized scheme and give some examples. In Section 4, we present some general results about smoothness analysis of proposed schemes and make analysis of some schemes. Some important properties of the schemes are discussed in Section 5. Applications, comparison and summary are given in Section 6.

2 Preliminary Results

A general form of univariate *b*-ary non-stationary subdivision scheme *S* which maps a polygon $f^k = \{f_i^k\}_{i\in Z}$ is defined by

$$f_{bi+\alpha}^{k+1} = \sum_{j=0}^{m} a_{bj+\alpha}^k f_{i+j}^k,\ \ \alpha = 0, 1, \dots, b-1. \tag{2.1}$$

The set $a^k = \{a_i^k : i\ \in\ Z\}$ is called the k^{th} level mask of the scheme. If the mask is dependent on the subdivision level then the subdivision scheme is termed as non-stationary otherwise it is stationary.

Theorem 2.1. Two subdivision schemes $\{S_{u^k}\}$ and $\{S_{v^k}\}$ are asymptotically equivalent if

$$\sum_{k=1}^{\infty} \left\| S_{u^k} - S_{v^k} \right\|_{\infty} < \infty,$$

where

$$\left\| S_{u^k} \right\|_{\infty} = \max\left\{ \sum_{\alpha \in Z} \left| a_{b\alpha}^{k} \right|, \sum_{\alpha \in Z} \left| a_{b\alpha+1}^{k} \right|, \ldots, \sum_{\alpha \in Z} \left| a_{b\alpha+(b-1)}^{k} \right| \right\}.$$

The idea behind asymptotic equivalence was presented by Dyn and Levin [15]. The proof of following theorem follows exactly similar to the proof of the theorem given in (Theorem 8, [15]).

Theorem 2.2. Let $\{S_k\}$ and $\{S\}$ be two b-ary subdivision schemes having finite masks of the same support. Suppose $\{S_k\}$ is non-stationary and $\{S\}$ is stationary scheme. If $\{S\}$ is C^m and $\sum_{k=1}^{\infty} b^{mk} \|S_k - S\|_{\infty} < \infty$ then the non-stationary scheme $\{S_k\}$ is C^m.

3 Generalized *4*-Point *b*-ary Scheme

In this section, we present 4-point approximating non-stationary subdivision scheme of even-arity. These schemes are constructed by interpolation with the space $\xi = span\{1, \sin\beta x, \cos\beta x\}$, for some $\beta, 0 < \beta < \pi/2$.

Let us assume that we have a data set $S = \left\{ \left(x_j , p(x_j) \right) : \ j = \ 0, 1, 2, 3 \right\}$. Consider the function

$$L(x) = \sum_{j=0}^{3} p(x_j) \cos \beta \left(\frac{x - x_j}{2} \right) L_j(x),$$

where

$$L_j(x) = \prod_{k=0, k\neq j}^{3} \frac{\sin \beta \left(\frac{x - x_k}{2} \right)}{\sin \beta \left(\frac{x_j - x_k}{2} \right)},$$

It is known [16] that $L(x)$ is not the unique function in ξ which interpolates S. We label the function $L(x)$, a Lagrange like interpolant of the above data.

We denote the function $L_j(x), j = 0, 1, 2, 3$ by $L_j^N(\mathrm{x})$ correspond to the data $\mathrm{N} = \{ x_j = \mathrm{j} - 1 :$ $: \mathrm{j} = 0, 1, 2, 3$.

Then

$$L_j(x) = \prod_{k=0, k\neq j}^{3} \frac{\sin \beta \left(\frac{x - x_k}{2} \right)}{\sin \beta \left(\frac{x_j - x_k}{2} \right)}, \quad x = \frac{q}{2b}, \tag{3.1}$$

where $q = 1, 3, 5, \ldots, b - 1$ (any odd integer) for $b \geq 2$ (any even integer).

A general *4*-point *b*-ary approximating non-stationary subdivision scheme associated with the interpolation with the space ξ is given as follows:

Given the initial control points $f_i^0 \in R$ and for β such that $0 < \beta < \pi/2$, the control points f_i^{k+1} at level $k+1$ are given by the following recursive algorithm:

$$\begin{cases} f_{bi+v}^{k+1} = -\eta_{v+1,0}^k f_{i-1}^k + \eta_{v+1,1}^k f_i^k + \eta_{v+1,2}^k f_{i+1}^k - \eta_{v+1,3}^k f_{i+2}^k, \\ f_{bi+w}^{k+1} = -\eta_{b-w,3}^k f_{i-1}^k + \eta_{b-w,2}^k f_i^k + \eta_{b-w,1}^k f_{i+1}^k - \eta_{b-w,0}^k f_{i+2}^k, \end{cases} \tag{3.2}$$

where $v = 0, 1, \dots, t-1,\ w = t, t+1, \dots, b-1,\ t = \frac{b}{2},\quad p = 2b - q, y = 2b + q,\quad z = 2b + p, q = 1, 3, \dots, b-1$, (any odd integer) for $b \geq 2$ (any even integer) and

$$\eta_{q,0}^k = \frac{\cos\left(\frac{y\beta}{4.b^{k+1}}\right)\sin\left(\frac{q\beta}{4.b^{k+1}}\right)\sin\left(\frac{p\beta}{4.b^{k+1}}\right)\sin\left(\frac{z\beta}{4.b^{k+1}}\right)}{\sin\left(\frac{\beta}{2.b^k}\right)\sin\left(\frac{2\beta}{2.b^k}\right)\sin\left(\frac{3\beta}{2.b^k}\right)},$$

$$\eta_{q,1}^k = \frac{\cos\left(\frac{q\beta}{4.b^{k+1}}\right)\sin\left(\frac{p\beta}{4.b^{k+1}}\right)\sin\left(\frac{y\beta}{4.b^{k+1}}\right)\sin\left(\frac{z\beta}{4.b^{k+1}}\right)}{\sin^2\left(\frac{\beta}{2.b^k}\right)\sin\left(\frac{2\beta}{2.b^k}\right)},$$

$$\eta_{q,2}^k = \frac{\cos\left(\frac{p\beta}{4.b^{k+1}}\right)\sin\left(\frac{q\beta}{4.b^{k+1}}\right)\sin\left(\frac{y\beta}{4.b^{k+1}}\right)\sin\left(\frac{z\beta}{4.b^{k+1}}\right)}{\sin^2\left(\frac{\beta}{2.b^k}\right)\sin\left(\frac{2\beta}{2.b^k}\right)},$$

$$\eta_{q,3}^k = \frac{\cos\left(\frac{z\beta}{4.b^{k+1}}\right)\sin\left(\frac{q\beta}{4.b^{k+1}}\right)\sin\left(\frac{p\beta}{4.b^{k+1}}\right)\sin\left(\frac{y\beta}{4.b^{k+1}}\right)}{\sin\left(\frac{\beta}{2.b^k}\right)\sin\left(\frac{2\beta}{2.b^k}\right)\sin\left(\frac{3\beta}{2.b^k}\right)}.$$

3.1 Some Examples of 4-Point b-ary Scheme

Here we derive *4*-point binary and *4*-point quaternary schemes from general *4*-point *b*-ary scheme (3.2). Similarly, we can easily derive other higher arity *4*-point schemes.

3.1.1 4-point binary scheme

By substituting $b = 2$ in (3.2), we get following *4*-point binary approximating non-stationary scheme

$$\begin{cases} f_{2i}^{k+1} = -\eta_{1,0}^k f_{i-1}^k + \eta_{1,1}^k f_i^k + \eta_{1,2}^k f_{i+1}^k - \eta_{1,3}^k f_{i+2}^k, \\ f_{2i+1}^{k+1} = -\eta_{1,3}^k f_{i-1}^k + \eta_{1,2}^k f_i^k + \eta_{1,1}^k f_{i+1}^k - \eta_{1,0}^k f_{i+2}^k, \end{cases} \tag{3.3}$$

where

$$\eta_{1,0}^k = \frac{\cos\left(\frac{5\beta}{4.2^{k+1}}\right)\sin\left(\frac{\beta}{4.2^{k+1}}\right)\sin\left(\frac{3\beta}{4.2^{k+1}}\right)\sin\left(\frac{7\beta}{4.2^{k+1}}\right)}{\sin\left(\frac{\beta}{2.2^k}\right)\sin\left(\frac{2\beta}{2.2^k}\right)\sin\left(\frac{3\beta}{2.2^k}\right)},$$

$$\eta_{1,1}^k = \frac{\cos\left(\frac{\beta}{4.2^{k+1}}\right)\sin\left(\frac{5\beta}{4.2^{k+1}}\right)\sin\left(\frac{3\beta}{4.2^{k+1}}\right)\sin\left(\frac{7\beta}{4.2^{k+1}}\right)}{\sin^2\left(\frac{\beta}{2.2^k}\right)\sin\left(\frac{2\beta}{2.2^k}\right)},$$

$$\eta_{1,2}^k = \frac{\cos\left(\frac{3\beta}{4.2^{k+1}}\right)\sin\left(\frac{\beta}{4.2^{k+1}}\right)\sin\left(\frac{5\beta}{4.2^{k+1}}\right)\sin\left(\frac{7\beta}{4.2^{k+1}}\right)}{\sin^2\left(\frac{\beta}{2.2^k}\right)\sin\left(\frac{2\beta}{2.2^k}\right)},$$

$$\eta_{1,3}^k = \frac{\cos\left(\frac{7\beta}{4.2^{k+1}}\right)\sin\left(\frac{5\beta}{4.2^{k+1}}\right)\sin\left(\frac{3\beta}{4.2^{k+1}}\right)\sin\left(\frac{\beta}{4.2^{k+1}}\right)}{\sin\left(\frac{\beta}{2.2^k}\right)\sin\left(\frac{2\beta}{2.2^k}\right)\sin\left(\frac{3\beta}{2.2^k}\right)}.$$

3.1.2 4-point quaternary scheme

By substituting $b = 4$ in (3.2), we get following *4*-point quaternary approximating non-stationary scheme

$$\begin{cases} f_{4i}^{k+1} = -\eta_{1,0}^k f_{i-1}^k + \eta_{1,1}^k f_i^k + \eta_{1,2}^k f_{i+1}^k - \eta_{1,3}^k f_{i+2}^k, \\ f_{4i+1}^{k+1} = -\eta_{2,0}^k f_{i-1}^k + \eta_{2,1}^k f_i^k + \eta_{2,2}^k f_{i+1}^k - \eta_{2,3}^k f_{i+2}^k, \\ f_{4i+2}^{k+1} = -\eta_{2,3}^k f_{i-1}^k + \eta_{2,2}^k f_i^k + \eta_{2,1}^k f_{i+1}^k - \eta_{2,0}^k f_{i+2}^k, \\ f_{4i+3}^{k+1} = -\eta_{1,3}^k f_{i-1}^k + \eta_{1,2}^k f_i^k + \eta_{1,1}^k f_{i+1}^k - \eta_{1,0}^k f_{i+2}^k, \end{cases} \tag{3.4}$$

where

$$\eta_{1,0}^k = \frac{\cos\left(\frac{9\beta}{4.4^{k+1}}\right)\sin\left(\frac{\beta}{4.4^{k+1}}\right)\sin\left(\frac{7\beta}{4.4^{k+1}}\right)\sin\left(\frac{15\beta}{4.4^{k+1}}\right)}{\sin\left(\frac{\beta}{2.4^k}\right)\sin\left(\frac{2\beta}{2.4^k}\right)\sin\left(\frac{3\beta}{2.4^k}\right)},$$

$$\eta_{1,1}^k = \frac{\cos\left(\frac{\beta}{4.4^{k+1}}\right)\sin\left(\frac{9\beta}{4.4^{k+1}}\right)\sin\left(\frac{7\beta}{4.4^{k+1}}\right)\sin\left(\frac{15\beta}{4.4^{k+1}}\right)}{\sin^2\left(\frac{\beta}{2.4^k}\right)\sin\left(\frac{2\beta}{2.4^k}\right)},$$

$$\eta_{1,2}^k = \frac{\cos\left(\frac{7\beta}{4.4^{k+1}}\right)\sin\left(\frac{\beta}{4.4^{k+1}}\right)\sin\left(\frac{9\beta}{4.4^{k+1}}\right)\sin\left(\frac{15\beta}{4.4^{k+1}}\right)}{\sin^2\left(\frac{\beta}{2.4^k}\right)\sin\left(\frac{2\beta}{2.4^k}\right)},$$

$$\eta_{1,3}^k = \frac{\cos\left(\frac{15\beta}{4.4^{k+1}}\right)\sin\left(\frac{\beta}{4.4^{k+1}}\right)\sin\left(\frac{7\beta}{4.4^{k+1}}\right)\sin\left(\frac{9\beta}{4.4^{k+1}}\right)}{\sin\left(\frac{\beta}{2.4^k}\right)\sin\left(\frac{2\beta}{2.4^k}\right)\sin\left(\frac{3\beta}{2.4^k}\right)},$$

$$\eta_{2,0}^{k}=\frac{\cos\left(\frac{11\beta}{4.4^{k+1}}\right)\sin\left(\frac{3\beta}{4.4^{k+1}}\right)\sin\left(\frac{5\beta}{4.4^{k+1}}\right)\sin\left(\frac{13\beta}{4.4^{k+1}}\right)}{\sin\left(\frac{\beta}{2.4^{k}}\right)\sin\left(\frac{2\beta}{2.4^{k}}\right)\sin\left(\frac{3\beta}{2.4^{k}}\right)},$$

$$\eta_{2,1}^{k}=\frac{\cos\left(\frac{3\beta}{4.4^{k+1}}\right)\sin\left(\frac{11\beta}{4.4^{k+1}}\right)\sin\left(\frac{5\beta}{4.4^{k+1}}\right)\sin\left(\frac{13\beta}{4.4^{k+1}}\right)}{\sin^{2}\left(\frac{\beta}{2.4^{k}}\right)\sin\left(\frac{2\beta}{2.4^{k}}\right)},$$

$$\eta_{2,2}^{k}=\frac{\cos\left(\frac{5\beta}{4.4^{k+1}}\right)\sin\left(\frac{11\beta}{4.4^{k+1}}\right)\sin\left(\frac{3\beta}{4.4^{k+1}}\right)\sin\left(\frac{13\beta}{4.4^{k+1}}\right)}{\sin^{2}\left(\frac{\beta}{2.4^{k}}\right)\sin\left(\frac{2\beta}{2.4^{k}}\right)},$$

$$\eta_{2,3}^{k}=\frac{\cos\left(\frac{13\beta}{4.4^{k+1}}\right)\sin\left(\frac{3\beta}{4.4^{k+1}}\right)\sin\left(\frac{5\beta}{4.4^{k+1}}\right)\sin\left(\frac{11\beta}{4.4^{k+1}}\right)}{\sin\left(\frac{\beta}{2.4^{k}}\right)\sin\left(\frac{2\beta}{2.4^{k}}\right)\sin\left(\frac{3\beta}{2.4^{k}}\right)}.$$

4 Continuity Analysis of *b*-ary Schemes

In this section, we present continuity analysis of *4*-point *b*-ary approximating non-stationary subdivision scheme. For the analysis of non-stationary schemes we use the notion of asymptotical equivalence [15]. In order to prove the convergence, we need some estimates of $\eta_{q,i}^{k}, i=0,1,2,3,$ which are given in the following two lemmas.

Lemma 4.1.

$$(i)\,\frac{qpz}{48b^{3}}\leq\eta_{q,0}^{k}\leq\frac{qpz}{48b^{3}\cos^{3}\left(\frac{3\beta}{2b^{k}}\right)},\qquad(ii)\;\frac{pzy}{16b^{3}}\leq\eta_{q,1}^{k}\leq\frac{pzy}{16b^{3}\cos^{3}\left(\frac{2\beta}{2b^{k}}\right)},$$

$$(iii)\,\frac{qzy}{16b^{3}}\leq\eta_{q,2}^{k}\leq\frac{qzy}{16b^{3}\cos^{3}\left(\frac{2\beta}{2b^{k}}\right)},\qquad(ii)\;\frac{qpy}{48b^{3}}\leq\eta_{q,3}^{k}\leq\frac{qpy}{48b^{3}\cos^{3}\left(\frac{3\beta}{2b^{k}}\right)},$$

where $p=2b-q,\ y=2b+q,\ z=2b+p\ and\ q=1,\ 3,\dots,b-1,$ (any odd integer) for $b\geq 2$ (any even integer).

Proof. Since

$$\eta_{q,0}^{k}\geq\frac{\frac{1}{2}\frac{q\beta}{4b^{k+1}}\frac{p\beta}{4b^{k+1}}\frac{z\beta}{4b^{k+1}}}{\frac{\beta}{2b^{k}}\frac{2\beta}{2b^{k}}\frac{3\beta}{2b^{k}}}=\frac{qpz}{48b^{3}}$$

and

$$\eta_{q,0}^{k}\leq\frac{\frac{q\beta}{4b^{k+1}}\frac{p\beta}{4b^{k+1}}\frac{z\beta}{4b^{k+1}}}{\frac{\beta}{2b^{k}}\cos\left(\frac{\beta}{2b^{k}}\right)\frac{2\beta}{2b^{k}}\cos\left(\frac{2\beta}{2b^{k}}\right)\frac{3\beta}{2b^{k}}\cos\left(\frac{3\beta}{2b^{k}}\right)}\leq\frac{qpz}{48b^{3}\cos\left(\frac{3\beta}{2b^{k}}\right)},$$

so the proof of (i) is completed. The proofs of (ii), (iii) and (iv) are similar to the proof of (i).

By substituting $b = 2$ and 4 in Lemma 4.1, we get following corollaries.

Corollary 4.2.

$$(i)\,\frac{7}{128} \leq \eta_{1,0}^{k} \leq \frac{7}{128\cos^3\left(\frac{3\beta}{2^{k+1}}\right)}, \quad (ii)\,\frac{105}{128} \leq \eta_{1,1}^{k} \leq \frac{105}{128\cos^3\left(\frac{2\beta}{2^{k+1}}\right)},$$
$$(iii)\,\frac{35}{128} \leq \eta_{1,2}^{k} \leq \frac{35}{128\cos^3\left(\frac{2\beta}{2^{k+1}}\right)}, \quad (ii)\,\frac{5}{128} \leq \eta_{1,3}^{k} \leq \frac{5}{128\cos^3\left(\frac{3\beta}{2^{k+1}}\right)}.$$

Remark 4.1. The scheme (3.3) is non-stationary counterpart of *4*-point binary C^2 scheme [17] with mask

$$\frac{1}{128}\{-5, -7, 35, 105, 105, 35, -7, -5\}, \qquad (4.1)$$

as by above corollary the mask of scheme (3.3) converges to the mask (4.1): $\eta_{1,0}^{k} \to \frac{7}{128}$, $\eta_{1,1}^{k} \to \frac{105}{128}$, $\eta_{1,2}^{k} \to \frac{35}{128}$ and $\eta_{1,3}^{k} \to \frac{5}{128}$, for $k \to \infty$.

Corollary 4.3.

$$(i)\,\frac{35}{1024} \leq \eta_{1,0}^{k} \leq \frac{35}{1024\cos^3\left(\frac{3\beta}{2.4^{k}}\right)}, \quad (ii)\,\frac{945}{1024} \leq \eta_{1,1}^{k} \leq \frac{945}{1024\cos^3\left(\frac{2\beta}{2.4^{k}}\right)},$$

$$(iii)\,\frac{135}{1024} \leq \eta_{1,2}^{k} \leq \frac{135}{1024\cos^3\left(\frac{2\beta}{2.4^{k}}\right)}, \quad (iv)\,\frac{21}{1024} \leq \eta_{1,3}^{k} \leq \frac{21}{1024\cos^3\left(\frac{3\beta}{2.4^{k}}\right)},$$

$$(v)\,\frac{65}{1024} \leq \eta_{2,0}^{k} \leq \frac{65}{1024\cos^3\left(\frac{3\beta}{2.4^{k}}\right)}, \quad (vi)\,\frac{715}{1024} \leq \eta_{2,1}^{k} \leq \frac{715}{1024\cos^3\left(\frac{2\beta}{2.4^{k}}\right)},$$

$$(vii)\,\frac{429}{1024} \leq \eta_{2,2}^{k} \leq \frac{429}{1024\cos^3\left(\frac{2\beta}{2.4^{k}}\right)}, \quad (viii)\,\frac{55}{1024} \leq \eta_{2,3}^{k} \leq \frac{55}{1024\cos^3\left(\frac{3\beta}{2.4^{k}}\right)}.$$

Remark 4.2. From the above corollary it is obvious that $\eta_{1,0}^{k} \to \frac{35}{1024}$, $\eta_{1,1}^{k} \to \frac{945}{1024}$, $\eta_{1,2}^{k} \to \frac{135}{1024}$, $\eta_{1,3}^{k} \to \frac{21}{1024}$, $\eta_{2,0}^{k} \to \frac{65}{1024}$, $\eta_{2,1}^{k} \to \frac{715}{1024}$, $\eta_{2,2}^{k} \to \frac{429}{1024}$ and $\eta_{2,3}^{k} \to \frac{55}{1024}$ as $k \to \infty$. It means that mask of the scheme (3.4) converges to the mask

$$\frac{1}{1024}\{-21, -55, -65, -35, 135, 429, 715, 945, 945, 715, 429, 135, -35, -65, -55, -21\}, \qquad (4.2)$$

of 4-point quaternary C^2 scheme [18] for $w = \frac{37}{16}$. So the scheme (3.4) is non-stationary counterpart of (4.2).

By using Lemma 4.1, we have the following lemma.

Lemma 4.4.

$$(i)\left|\eta_{q,0}^{k}-\frac{qpz}{48b^3}\right|\le\zeta_{q,0}\frac{1}{b^{2k}},\qquad (ii)\left|\eta_{q,1}^{k}-\frac{pzy}{16b^3}\right|\le\zeta_{q,1}\frac{1}{b^{2k}},$$
$$(iii)\left|\eta_{q,2}^{k}-\frac{qzy}{16b^3}\right|\le\zeta_{q,2}\frac{1}{b^{2k}},\qquad (iv)\left|\eta_{q,3}^{k}-\frac{qpy}{48b^3}\right|\le\zeta_{q,3}\frac{1}{b^{2k}},$$

where $p=2b-q,\ y=2b+q,\ \ z=2b+p\ and\ q=1,3,\ldots,b-1$, (any odd integer) for $b\ge 2$ (any even integer). The constants $\zeta_{q,0}$, $\zeta_{q,1}$, $\zeta_{q,2}$, and $\zeta_{q,3}$ are independent of k.

Proof. To prove (i), we have

$$\left|\eta_{q,0}^{k}-\frac{qpz}{48b^3}\right|\le\left|\frac{qpz}{48b^3\cos^3\left(\frac{3\beta}{2b^k}\right)}-\frac{qpz}{48b^3}\right|.$$

This implies for $0<\beta<\pi/2$

$$\left|\eta_{q,0}^{k}-\frac{qpz}{48b^3}\right|\le\frac{qpz}{48b^3}\left(\frac{1-\cos^3\left(\frac{3\beta}{2b^k}\right)}{\cos^3\left(\frac{3\beta}{2b^k}\right)}\right).$$

Again implies

$$\left|\eta_{q,0}^{k}-\frac{qpz}{48b^3}\right|\le\frac{qpz}{8b^3}\left(\frac{\sin^2\left(\frac{3\beta}{4b^k}\right)}{\cos^3(3\beta)}\right).$$

Finally, we get

$$\left|\eta_{q,0}^{k}-\frac{qpz}{48b^3}\right|\le\frac{1}{b^{2k}}\left(\frac{9\beta^2qpz}{128\,b^3\cos^3(3\beta)}\right)=\zeta_{q,0}\frac{1}{b^{2k}},$$

where $\zeta_{q,0}=\frac{9\beta^2qpz}{128\,b^3\cos^3(3\beta)}$ is independent of k. The proofs of (ii), (iii) and (iv) are similar to the proof of (i).

By substituting $b=2$ and *4* in Lemma 4.4, we get following corollaries.

Corollary 4.5.

$$(i)\left|\eta_{1,0}^{k}-\frac{7}{128}\right|\le\zeta_{1,0}\frac{1}{2^{2k}},\qquad (ii)\left|\eta_{1,1}^{k}-\frac{105}{128}\right|\le\zeta_{1,1}\frac{1}{2^{2k}},$$
$$(iii)\left|\eta_{1,2}^{k}-\frac{35}{128}\right|\le\zeta_{1,2}\frac{1}{2^{2k}},\qquad (iv)\left|\eta_{1,3}^{k}-\frac{5}{128}\right|\le\zeta_{1,3}\frac{1}{2^{2k}}.$$

Corollary 4.6.

$$(i)\ \left|\eta_{1,0}^{k}-\frac{35}{1024}\right|\leq\zeta_{1,0}\frac{1}{4^{2k}},\qquad (ii)\ \left|\eta_{1,1}^{k}-\frac{945}{1024}\right|\leq\zeta_{1,1}\frac{1}{4^{2k}},$$
$$(iii)\ \left|\eta_{1,2}^{k}-\frac{135}{1024}\right|\leq\zeta_{1,2}\frac{1}{4^{2k}},\qquad (iv)\ \left|\eta_{1,3}^{k}-\frac{21}{1024}\right|\leq\zeta_{1,3}\frac{1}{4^{2k}},$$
$$(v)\ \left|\eta_{2,0}^{k}-\frac{65}{1024}\right|\leq\zeta_{2,0}\frac{1}{4^{2k}},\qquad (vi)\ \left|\eta_{2,1}^{k}-\frac{715}{1024}\right|\leq\zeta_{2,1}\frac{1}{4^{2k}},$$
$$(vii)\ \left|\eta_{2,2}^{k}-\frac{429}{1024}\right|\leq\zeta_{2,2}\frac{1}{4^{2k}},\qquad (viii)\ \left|\eta_{2,3}^{k}-\frac{55}{1024}\right|\leq\zeta_{2,3}\frac{1}{4^{2k}}.$$

Theorem 4.7. The proposed *4*-point binary non-stationary scheme (3.3) is C^2.
Proof. Let S_{u^k} and S_u denote the schemes (3.3) and (4.1) having finite mask of same support respectively.

We claim that

$$\sum_{k=1}^{\infty} 2^{2k}\left\|S_{u^k}-S_u\right\|_{\infty}<\infty,$$

where

$$\left\|S_{u^k}-S_u\right\|_{\infty}=\max\left\{\sum_{j\in Z}\left|u_{i-2j}^{k}-u_{i-2j}\right|,\ i=0,1,2,3\right\}.$$

From the schemes (3.3) and (4.1), we have

$$\sum_{k=0}^{\infty} 2^{2k}\left\|S_{u^k}-S_u\right\|_{\infty}=\sum_{k=0}^{\infty} 2^{2k}\left(\left|u_{-4}^{k}-u_{-4}\right|+\left|u_{-2}^{k}-u_{-2}\right|+\left|u_{0}^{k}-u_{0}\right|+\left|u_{2}^{k}-u_{2}\right|\right).$$

This implies that

$$\sum_{k=0}^{\infty} 2^{2k}\left\|S_{u^k}-S_u\right\|_{\infty}=\sum_{k=0}^{\infty} 2^{2k}\left(\left|\eta_{1,3}^{k}-\frac{5}{128}\right|+\left|\eta_{1,2}^{k}-\frac{35}{128}\right|+\left|\eta_{1,1}^{k}-\frac{105}{128}\right|+\left|\eta_{1,0}^{k}-\frac{7}{128}\right|\right).$$

From (i) of Corollary 4.5, we have

$$\sum_{k=0}^{\infty} 2^{2k}\left|\eta_{1,0}^{k}-\frac{7}{128}\right|\leq\sum_{k=0}^{\infty} 2^{2k}\zeta_{1,0}\frac{1}{2^{2k}}<\infty.$$

In the same way by using (ii), (iii) and (iv) of Corollary 4.5, we can easily show that

$\sum_{k=0}^{\infty} 2^{2k}\left|\eta_{1,1}^{k}-\frac{105}{128}\right|<\infty, \sum_{k=0}^{\infty} 2^{2k}\left|\eta_{1,2}^{k}-\frac{35}{128}\right|<\infty$ and $\sum_{k=0}^{\infty} 2^{2k}\left|\eta_{1,3}^{k}-\frac{5}{128}\right|<\infty.$

Therefore, we have $\sum_{k=1}^{\infty} 2^{2k}\left\|S_{u^k}-S_u\right\|_{\infty}<\infty$, which means that both schemes (3.3) and (4.1) are asymptotically equivalent. Since (4.1) is C^2 so by Theorem 2.2, (3.3) is also C^2.

By using Corollary 4.6, we get following theorem.

Theorem 4.8. The proposed 4-point quaternary non-stationary scheme (3.4) is C^2.

***Proof*.** Proof of this theorem follows the proof of Theorem 4.7.

5 Properties of the Subdivision Schemes

In this section, we discuss some important properties of the proposed schemes.

5.1 Affine Invariance Property

In the following proposition, we show that the scheme (3.2) holds affine invariance property.

Proposition 5.1. The scheme (3.2) satisfies affine invariance property i.e. $-\eta_{q,0}^k + \eta_{q,1}^k + \eta_{q,2}^k - \eta_{q,3}^k = 1$.

Proof. Since

$$\eta_{q,0}^k + \eta_{q,3}^k = \frac{\sin\left(\frac{q\beta}{4.b^{k+1}}\right)\sin\left(\frac{p\beta}{4.b^{k+1}}\right)\sin\left(\frac{6b\beta}{4.b^{k+1}}\right)}{\sin\left(\frac{\beta}{2.b^k}\right)\sin\left(\frac{2\beta}{2.b^k}\right)\sin\left(\frac{3\beta}{2.b^k}\right)},$$

then this implies

$$\eta_{q,0}^k + \eta_{q,3}^k = \frac{\sin\left(\frac{q\beta}{4.b^{k+1}}\right)\sin\left(\frac{p\beta}{4.b^{k+1}}\right)}{\sin\left(\frac{\beta}{2.b^k}\right)\sin\left(\frac{2\beta}{2.b^k}\right)}. \tag{5.1}$$

Similarly, we have

$$\eta_{q,1}^k + \eta_{q,2}^k = \frac{\sin\left(\frac{z\beta}{4.b^{k+1}}\right)\sin\left(\frac{y\beta}{4.b^{k+1}}\right)}{\sin\left(\frac{\beta}{2.b^k}\right)\sin\left(\frac{2\beta}{2.b^k}\right)}. \tag{5.2}$$

By subtracting (5.1) and (5.2), we have

$$-\eta_{q,0}^k + \eta_{q,1}^k + \eta_{q,2}^k - \eta_{q,3}^k = \frac{1}{2}\left(\frac{\cos\left(\frac{2b\beta}{4.b^{k+1}}\right) - \cos\left(\frac{6b\beta}{4.b^{k+1}}\right)}{\sin\left(\frac{\beta}{2.b^k}\right)\sin\left(\frac{2\beta}{2.b^k}\right)}\right).$$

This implies that

$$-\eta_{q,0}^k + \eta_{q,1}^k + \eta_{q,2}^k - \eta_{q,3}^k = \frac{1}{2}\left(\frac{-2\sin\left(\frac{-\beta}{2.b^k}\right)\sin\left(\frac{2\beta}{2.b^k}\right)}{\sin\left(\frac{\beta}{2.b^k}\right)\sin\left(\frac{2\beta}{2.b^k}\right)}\right).$$

Thus we have

$$-\eta_{q,0}^{k} + \eta_{q,1}^{k} + \eta_{q,2}^{k} - \eta_{q,3}^{k} = 1.$$

Corollary 5.2. The schemes (3.3) and (3.4) satisfy affine invariance property.

5.2 Basic Limit Function

The basic limit function B of a scheme is defined as the limit function of the scheme for the data $f_0^i = \delta_{i,0}$, where $\delta_{i,0}$ is Kronecker delta. By Theorems 4.7 and 4.8 it follows that the basic functions defined by the proposed schemes (3.3) and (3.4) generate C^2-continues limit curves. These functions are shown in Fig. 1.

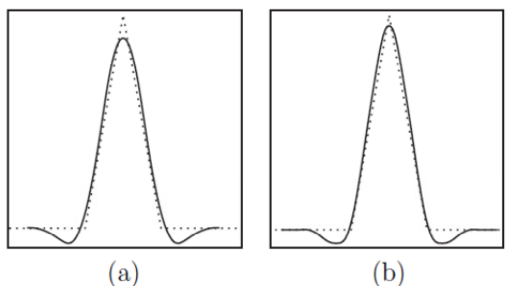

Fig. 1. (a) and (b) present basic function of proposed schemes (3.3) and (3.4) respectively.

Now we derive a general relation to calculate support width of 4-point b-ary scheme. We figure out that as we increase arity of 4-point b-ary scheme the support width decreases, i.e. for 4-point b-ary scheme arity and support width are reciprocal to each other.

Proposition 5.3. The basic function B defined by proposed scheme (3.2) has support width $\lambda = \frac{4b-1}{b-1}$, which implies that it vanishes outside the interval $\left[-\frac{4b-1}{2(b-1)}, \frac{4b-1}{2(b-1)}\right]$.

Proof. Since the basic function B is the limit function of the scheme (3.2), its support width λ can be determined by computing how far the effect of the non-zero vertex f_0^0 will propagate along by. As the mask of the scheme is a $4b$-long sequence by centering it on that vertex, the distances to the last of its left and right non-zero coefficients are equal to $2b$ and $2b-1$ respectively. At the first subdivision step we see that the vertices on the left and right sides of f_0^1 at $\frac{2b}{b}$ and $\frac{2b-1}{b}$ are the furthest nonzero new vertices. At each refinement, the distance on both sides is reduced by the factor $\frac{1}{b}$. At the next step of the scheme this will propagate along by $\frac{2b}{b^2}$ on left and $\frac{2b-1}{b^2}$ on right. Hence after k subdivision steps the furthest non-zero vertex on the left will be at

$$2b\left(\frac{1}{b} + \frac{1}{b^2} + \frac{1}{b^3} + \cdots + \frac{1}{b^k}\right) = \frac{2b}{b}\left(\sum_{j=0}^{k-1} \frac{1}{b^j}\right)$$

And

$$(2b-1)\left(\frac{1}{b}+\frac{1}{b^2}+\frac{1}{b^3}+\cdots+\frac{1}{b^k}\right)=\frac{(2b-1)}{b}\left(\sum_{j=0}^{k-1}\frac{1}{b^j}\right).$$

Since $\frac{1}{b}<1$, the geometric sequence can be summed to give the extended distance on each side and we conclude that, in the limit, the total influence of the original non-zero vertex will propagate along by

$$\lambda=\frac{2b}{b}\left(\sum_{j=0}^{k-1}\frac{1}{b^j}\right)+\frac{(2b-1)}{b}\left(\sum_{j=0}^{k-1}\frac{1}{b^j}\right)=\frac{4b-1}{b}\left(\frac{1}{1-\frac{1}{b}}\right)=\frac{4b-1}{b-1}.$$

Proposition 5.4. The basic function of the scheme (3.3) has support width $\lambda=7$, which implies that it vanishes outside the interval $\left[-\frac{7}{2},\frac{7}{2}\right]$.

Proposition 5.5. The basic function of the scheme (3.4) has support width $\lambda=5$, which implies that it vanishes outside the interval $\left[-\frac{5}{2},\frac{5}{2}\right]$.

Proposition 5.6. The basic function B defined by the proposed scheme (3.3) is symmetric about Y-axis.

Proof. Let us denote the set $G_k:=\left\{\frac{i}{2^k}\mid i\in Z\right\}$ such that the restriction of the basic function B to G_k satisfies $B\left(\frac{i}{2^k}\right)=f_i^k$, for all $i\in Z$ and we will use mathematical induction on *k* to prove this. First of all we note that $B(i)=f_i^0=f_{-i}^0=B(-i)$ for all $i\in Z$ and thus $B\left(\frac{i}{2^k}\right)=B\left(\frac{-i}{2^k}\right)=f_{-i}^k, \forall\, i\in Z, k=0$.

Now we assume that $B\left(\frac{i}{2^k}\right)=B\left(\frac{-i}{2^k}\right), \forall\, i\in Z$, then it follows that $f_i^k=B\left(\frac{i}{2^k}\right)=B\left(\frac{-i}{2^k}\right)=f_{-i}^k, \forall\, i\in Z$.

Therefore

$$B\left(\frac{2i}{2^{k+1}}\right)=-\eta_{1,0}^k f_{i-1}^k+\eta_{1,1}^k f_i^k+\eta_{1,2}^k f_{i+1}^k-\eta_{1,3}^k f_{i+2}^k.$$

This implies that

$$B\left(\frac{2i}{2^{k+1}}\right)=-\eta_{1,0}^k B\left(\frac{i-1}{2^k}\right)+\eta_{1,1}^k B\left(\frac{i}{2^k}\right)+\eta_{1,2}^k B\left(\frac{i+1}{2^k}\right)-\eta_{1,3}^k B\left(\frac{i+2}{2^k}\right).$$

So we have

$$B\left(\frac{2i}{2^{k+1}}\right)=-\eta_{1,0}^k B\left(\frac{-i+1}{2^k}\right)+\eta_{1,1}^k B\left(\frac{-i}{2^k}\right)+\eta_{1,2}^k B\left(\frac{-i-1}{2^k}\right)-\eta_{1,3}^k B\left(\frac{-i-2}{2^k}\right).$$

Thus we have

$$B\left(\frac{2i}{2^{k+1}}\right) = -\eta_{1,0}^{k} f_{-i+1}^{k} + \eta_{1,1}^{k} f_{-i}^{k} + \eta_{1,2}^{k} f_{-i-1}^{k} - \eta_{1,3}^{k} f_{-i-2}^{k} = f_{-2i}^{k+1} = B\left(\frac{-2i}{2^{k+1}}\right).$$

Similarly, we can easily show that

$$B\left(\frac{2i+1}{2^{k+1}}\right) = B\left(-\frac{2i+1}{2^{k+1}}\right).$$

Consequently $B\left(\frac{i}{2^k}\right) = B\left(\frac{-i}{2^k}\right), \forall\, i \in Z$ and $k \in Z_+$. As a result from continuity of B we have that $B(x) = B(-x), \forall\, x \in R$, which shows that the basic function B defined by the proposed *4*-point binary scheme (3.3) is symmetric about Y-axis.

In the same way, we can easily show that

Proposition 5.7. The basic limit function B defined by the proposed *4*-point quaternary scheme (3.4) is symmetric about Y-axis.

6 Applications, Comparison and Summary

In this section, we demonstrate visual performance of some of the proposed schemes by several examples. Comparison with existing binary and ternary non-stationary schemes and brief summary of work done so far is also included in this section.

Fig. 2 shows smooth curves which approximate set of given points. The control polygons are drawn by dashed lines and the smooth curves by full lines. Limit curves presented in 2(a)-2(c) are obtained by proposed scheme (3.3) after four iterations while limit curves shown in 2(d)-2(f) are obtained by proposed scheme (3.4) after two iterations.

In Table 1 we give brief comparison of *b*-ary schemes with some existing schemes. It is shown that the continuity of *b*-ary schemes is greater than the other existing binary and ternary schemes. Also as we increase the arity of *b*-ary schemes, support of the schemes decreases.

A generalized 4-point *b*-ary non-stationary approximating subdivision scheme is presented using trigonometric Lagrange polynomial. Asymptotic equivalence technique is used for continuity analysis of proposed scheme. It is also shown that proposed non-stationary schemes are counterpart of celebrated stationary schemes. Some important properties of proposed scheme like affine invariance, support and symmetry of basic limit function have been discussed. An explicit formula to calculate support width of basic limit function is established. We deduced that arity and support width of *4*-point *b*-ary scheme are reciprocal to each other. Visual performance of proposed scheme is shown by several examples.

Table 1. Comparison of b-ary schemes

Scheme	Type	Continuity	Support

2-point Binary [2]	Approximating	1	3
3-point Binary [4]	Approximating	1	5
4-point Binary [2]	Interpolating	1	6
3-point Ternary [2]	Interpolating	1	4
Proposed 4-point Binary	Approximating	2	7
Proposed 4-point Quaternary	Approximating	2	5

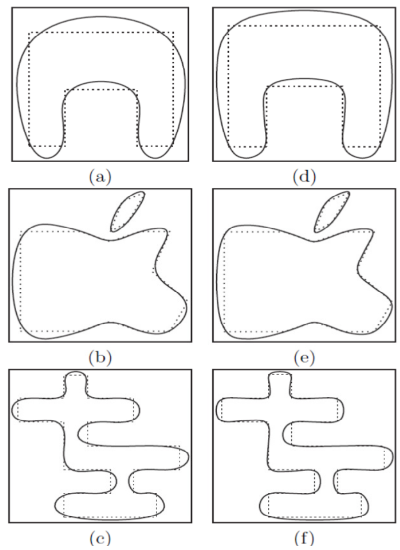

Fig. 2. (a)-(c) Present limit curve of the scheme (3.3) after 4th subdivision level and (d)-(f) present limit curve of the scheme (3.4) after 2nd subdivision level.

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Competing Interests

Authors have declared that no competing interests exist.

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