American Journal of Computational Mathematics, 2023, 13, 153-160
https://www.scirp.org/journal/ajcm
ISSN Online: 2161-1211
ISSN Print: 2161-1203

On the Numerical Solution of Singular Integral Equation with Degenerate Kernel Using Laguerre Polynomials

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How to cite this paper: Alroogy, K.S. and Aigo, M.A. (2023) On the Numerical Solution of Singular Integral Equation with Degenerate Kernel Using Laguerre Polynomials. *American Journal of Computational Mathematics*, **13**, 153-160.
https://doi.org/10.4236/ajcm.2023.131007

Received: December 13, 2022
Accepted: March 20, 2023
Published: March 23, 2023

Abstract

In this paper, we derive a simple and efficient matrix formulation using Laguerre polynomials to solve the singular integral equation with degenerate kernel. This method is based on replacement of the unknown function by truncated series of well known Laguerre expansion of functions. This leads to a system of algebraic equations with Laguerre coefficients. Thus, by solving the matrix equation, the coefficients are obtained. Some numerical examples are included to demonstrate the validity and applicability of the proposed method.

Keywords

Singular Integral Equation, Projection Method, Galerkin Method, Laguerre Polynomials

1. Introduction

Recent years, there has been a growing interest in the Fredholm and Volterra integral equations. This is an important branch of modern mathematics and arises frequently in many applied areas which include engineering, mechanics, physics, chemistry, astronomy, biology [1] [2]. There are several methods for approximating the solution of linear and non-linear integral equations [3]-[8]. We consider the singular integral equation of the second kind with smooth kernel:

$$\varphi(x) = f(x) + \lambda \int_0^\infty k(x,t)\varphi(t)\,\mathrm{d}t \tag{1}$$

where $f(x)$ is a continuous function for $x \geq 0$ and the kernel $k(x,t)$ is a function defined on the domain $D=\{(x,t): x \geq 0, t < \infty\}$ and $\varphi(x)$ is the unknown function that will be determined. In [9], Laguerre polynomials are used to derive numerical Solutions of Volterra integral Equations. In [10], repeated Simpson's and Trapezoidal quadrature rule was used to solve the linear Volterra Integral equations of the second kind. Since the integral equation is called singular when one or both limits of integration become infinite or when the kernel becomes infinite at one or more points in the domain of the integration. In our study, we are interested in the case where one limits of integration become infinite. For solving singular integral equations, many methods with enough accuracy and efficiency have been used before much research, see [8]-[13]. In [14], the author uses Toeplitz matrices method as numerical method to solve a singular integral equation, where many definite integrals cannot be computed in closed form, and must be approximated numerically. In [1], the orthogonal polynomials are used to solve numerically Nonlinear Volterra Fredholm Integral Equations. In [13], the Legendre and Chebyshev collocation method is presented to solve numerically the Voltterra-Fredholm Integral Equations with singular kernel. In this paper, we use numerical technique based on projection method, to reduce the singular integral Equations to a linear system of algebraic equations which will be solved using Gauss elimination or iterative methods. The paper is organized as follows. In section 2, we recall some properties related to Laguerre polynomials. In section 3, a system of algebraic equations will be presented based on Laguerre polynomials. In Section 4, we present a strategy to compute the exact solution for a singular integral equation with degenerate kernel. In section 5, we give a practical example to certify the validity of the proposed technique and then we conclude.

2. Laguerre Method

Sequences of orthogonal polynomials appear frequently used as applications in mathematics, mathematical physics, engineering and computer science, in particular during the resolution of partial differential equations (Laplace, Schrödinger) by the method of separation of variables, also these polynomials can be used to solve integral equations of first and second kind [1] [11]. One of the most common set of orthogonal polynomials is the Laguerre polynomials. Many families of orthogonal polynomials are known, which have in common a certain number of simple properties. The Laguerre differential equation is given by

$$xy''(x)+(1-x)y'(x)+ny(x)=0,\quad n=0,1,2,\cdots$$

The solutions of this equation are the Laguerre polynomials, expressed by the equation following differential:

$$L_n(x)=\mathrm{e}^{x}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^n\mathrm{e}^{-x}\right)$$

Then we get an approximation of the exactly integral, let say:

$$I_n(\Phi)=\int_0^{\infty}K(x,y)\varphi_n(y)\,\mathrm{d}y \tag{2}$$

is an approximation of the exact integral. This type of approximation must be chosen so that the integral (2) can be evaluated (either explicitly or by an efficient numerical technique). The functions $L_0(x), L_1(x), \cdots, L_n(x)$ will be called interpolating elements. In this paper, the interpolating function L_n will be assumed to be the interpolating polynomial

$$\varphi_n(x) = \sum_{j=0}^{n} c_j L_j(x) \tag{3}$$

where L_j are Laguerre polynomials of degree *j*, *n* is the number of Laguerre polynomials, and c_j are unknown parameters, to be determined.

3. System of Algebraic Equations

Consider the following systems

$$\{L_0(x), L_1(x), \cdots, L_n(x)\}$$

where

$$L_0(x) = 1, L_1(x) = 1 - x, \cdots, L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} x^k \qquad n \geq 2.$$

This system forms an orthonormal basis in $L^2(0, +\infty)$. In fact, we check that

$$\langle L_n(x), L_m(x) \rangle = \int_0^\infty L_n(x) L_m(x) \mathrm{e}^{-x} \mathrm{d}x = 0, \qquad m \neq n$$

$$\| L_n(x) \| = 1, \qquad n = 0, 1, 2, \cdots$$

The previous system is called the Laguerre polynomial system. To solve the integral Equation (1) we use the projection method. Using an approximation $\varphi_n(x)$ of the solution of Equation (1) which is a finite linear combination of orthogonal polynomials $(L_n(x))$ and also solution of the integral equation

$$\varphi_n(x) = f(x) + \lambda \int_0^\infty K(x,t) \varphi_n(t) \mathrm{d}t \tag{4}$$

By taking the linear combination of the Laguerre polynomials

$$\varphi_n(x) = \sum_{j=0}^{n} c_j L_j(x) \tag{5}$$

Substituting (5) in (4) we get

$$\sum_{j=0}^{n} c_j L_j(x) = f(x) + \lambda \int_0^\infty K(x,t) \left(\sum_{j=0}^{n} c_j L_j(t) \right) \mathrm{d}t \tag{6}$$

Let

$$H_j(x) = \int_0^\infty K(x,t) L_j(t) \mathrm{d}t$$

Then equation (6) can be written in the form

$$\sum_{j=0}^{n} c_j \left(L_j(x) - \lambda H_j(x) \right) = f(x) \tag{7}$$

By multiplying (7) by $L_i(x)$, we get

$$\sum_{j=0}^{n} c_j \langle L_j(x) - \lambda H_j(x), L_i(x) \rangle = \langle f(x), L_i(x) \rangle \tag{8}$$

Using the orthogonalization condition in Equation (8) we get

$$c_i - \lambda \sum_{j=0}^{n} c_j \left\langle H_j(x), L_i(x) \right\rangle = \left\langle f(x), L_i(x) \right\rangle, \qquad i = 0,1,\cdots,n \tag{9}$$

The system of Equation (9) has a unique solution if $D(\lambda) \neq 0$, where

$$D(\lambda) = \begin{vmatrix} 1-\lambda\left\langle H_0(x), L_0(x)\right\rangle & -\lambda\left\langle H_1(x), L_0(x)\right\rangle & & -\lambda\left\langle H_n(x), L_0(x)\right\rangle \\ -\lambda\left\langle H_0(x), L_1(x)\right\rangle & 1-\lambda\left\langle H_1(x), L_1(x)\right\rangle & & -\lambda\left\langle H_n(x), L_1(x)\right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda\left\langle H_0(x), L_n(x)\right\rangle & -\lambda\left\langle H_1(x), L_n(x)\right\rangle & & 1-\lambda\left\langle H_n(x), L_n(x)\right\rangle \end{vmatrix}$$

this makes it possible to determine the coefficients $(c_j)_{0 \le j \le n}$.

4. Exact Solution with Degenerate Kernel

Given a degenerate kernel $K(x,t) = P_1(x)P_2(t)$ then Equation (1) becomes

$$\varphi(x) = f(x) + \lambda P_1(x) \int_0^\infty P_2(t)\varphi(t)\,\mathrm{d}t \tag{10}$$

Let c be the number defined by

$$c = \int_0^\infty P_2(t)\varphi(t)\,\mathrm{d}t \tag{11}$$

Therefore

$$\varphi(x) = f(x) + \lambda c P_1(x) \tag{12}$$

substituting (12) into (11) gives

$$c = \frac{\int_0^\infty P_2(t) f(t)\,\mathrm{d}t}{1 - \lambda \int_0^\infty P_1(t) P_2(t)\,\mathrm{d}t} \tag{13}$$

from (13) and (12) we get:

$$\varphi(x) = f(x) + \lambda \frac{P_1(x) \int_0^\infty P_2(t) f(t)\,\mathrm{d}t}{1 - \lambda \int_0^\infty P_1(t) P_2(t)\,\mathrm{d}t} \tag{14}$$

provided that

$$\int_0^\infty P_1(t) P_2(t)\,\mathrm{d}t \neq \frac{1}{\lambda}$$

Note that, using some degenerate kernel one can compute exactly the integral to obtain an exact solution. Sometimes $\int_0^\infty K(x,y) L_n(y)\,\mathrm{d}y$ can not be evaluated exactly, for that one use quadrature rule to approximate the integral [14].

5. Numerical Examples

To confirm the validity, the accuracy and support our theoretical presentation of the proposed method, we give some computational examples. The computations, associated with the examples are performed by MATLAB. The system of algebraic system will be solved using Gauss elimination and iterative schemes will also be applied for large system.

5.1. First Example

In this example, we consider $\lambda = 1$ and $K(x,t) = e^{-t-x}, f(x) = x^3 - e^{-x}$; here we have $P_1(x) = e^{-x}$ and $P_2(t) = e^{-t}$ according to equation (14) we have

$$\varphi(x) = x^3 - e^{-x} + \frac{e^{-x}\int_0^\infty e^{-t}\left(t^3 - e^{-t}\right)dt}{1 - \int_0^\infty e^{-2t}dt}$$

Considering

$$I = \frac{\int_0^\infty e^{-t}\left(t^3 - e^{-t}\right)dt}{1 - \int_0^\infty e^{-2t}dt} = 11$$

so the exact solution of Equation (1) is $\varphi(x) = x^3 + 10e^{-x}$.

We used both Gauss elimination method, and SOR method iterations to solve the linear system. To plot the solution, we truncate the spacial domain to [0; 5]. The behavior of exact and numerical solution is presented in **Figure 1(a)**. The absolute error is depicted in **Figure 1(b)**. It is noticed that convergence to the exact solution need at least 20 Laguerre polynomials.

5.2. Second Example

Let $\lambda = 1$ and $K(x,t) = e^{-t^2-x^2}, f(x) = x^4 - \frac{3\sqrt{\pi}}{8}e^{-x^2}$ here we have $P_1(x) = e^{-x^2}$ and $P_2(t) = e^{-t^2}$ and by (14), the exaction solution is

$$\varphi(x) = x^4 - \frac{3\sqrt{\pi}}{8}e^{-x^2} + \frac{e^{-x^2}\int_0^\infty e^{-t^2}\left(t^4 - \frac{3\sqrt{\pi}}{8}e^{-t^2}\right)dt}{1 - \int_0^\infty e^{-2t^2}dt}$$

$$\varphi(x) = x^4 - \frac{3\sqrt{\pi}}{8}e^{-x^2} + e^{-x^2}\cdot\frac{\int_0^\infty t^4 e^{-t^2}dt - \frac{3\sqrt{\pi}}{8}\int_0^\infty e^{-2t^2}dt}{1 - \int_0^\infty e^{-2t^2}dt}$$

the Gaussian integral gives

$$\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$$

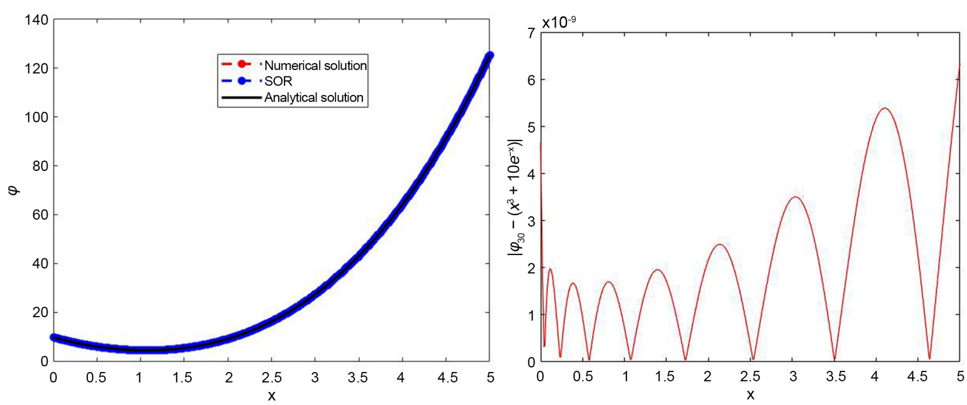

Figure 1. Analytical and numerical solution with $n = 30$. The absolute error $\left|\varphi_{30} - \left(x^3 + 10e^{-x}\right)\right|$.

by integration by parts, we have

$$\int_0^\infty t^4 e^{-t^2} dt = \frac{3\sqrt{\pi}}{8}$$

so the exact solution is $\varphi(x) = x^4$. For $n = 6$ the approximate solution

$$\varphi_6(x) = \sum_{j=0}^{6} c_j L_j(x)$$

with

$$L_0(x) = 1,$$

$$L_1(x) = 1 - x,$$

$$L_2(x) = \frac{1}{2}x^2 - 2x + 1,$$

$$L_3(x) = \frac{-1}{6}x^3 + \frac{3}{2}x^2 - 3x + 1,$$

$$L_4(x) = \frac{1}{24}x^4 - \frac{2}{3}x^3 + 3x^2 - 4x + 1,$$

$$L_5(x) = \frac{-1}{120}x^5 + \frac{5}{24}x^4 - \frac{5}{3}x^3 + 5x^2 - 5x + 1,$$

$$L_6(x) = \frac{1}{720}x^6 - \frac{1}{20}x^5 + \frac{5}{8}x^4 - \frac{10}{3}x^3 + \frac{15}{2}x^2 - 6x + 1,$$

the coefficients c_i are the solution of the system

$$c_i - \sum_{j=0}^{6} c_j \langle h_j(x), L_i(x) \rangle = \langle f(x), L_i(x) \rangle, \qquad i = 0, 1, \cdots, 6$$

$$h_j(x) = \int_0^\infty e^{-t^2 - x^2} L_j(t) dt$$

To plot the solution, we truncate the spacial domain to [0; 1]. The behavior of exact and numerical solution is depicted in Figure 2. Note that in example 2, the solution converge to the exact solution using only few Laguerre polynomail (n = 4).

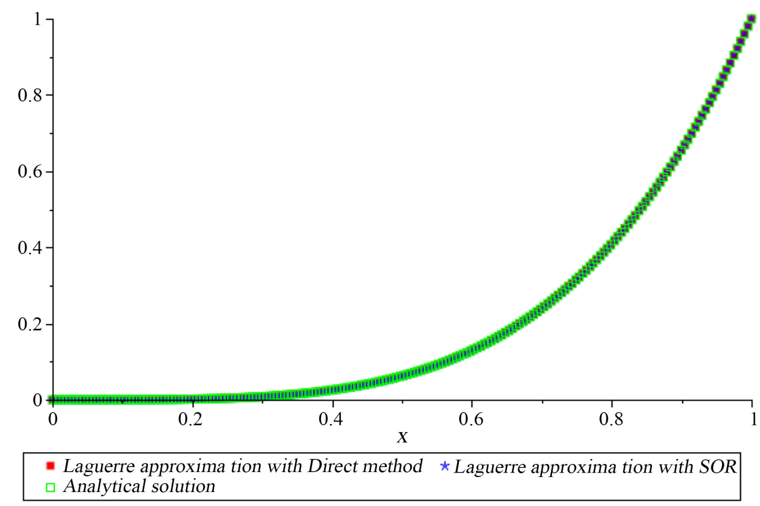

Figure 2. Example 2, case of n = 5. Comparison between exact solution and numerical solution.

6. Conclusion

A very simple and efficient method based on the Laguerre polynomial basis has been developed to solve singular integral equations. The result obtained confirms the strategy proposed. The method presented is tested and confirmed by two examples. A few numbers of Laguerre polynomials are needed to get convergence to the exact solution. Further, one can apply the proposed method to more general kernels or systems of integral equation.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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