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Three Dimensional Trans-Sasakian Manifold Admitting Quarter Symmetric Metric Connection

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Abstract

The object of this paper is to study quarter-symmetric metric connection on a trans-Sasakian manifold. Here we study locally ϕ-symmetric, ϕ-symmetric and concircular ϕ-symmetric trans-Sasakian manifold with respect to quarter symmetric metric connection and obtained some interesting results.

Keywords: Quarter symmetric metric connection, Trans-Sasakian manifold, ϕ-symmetry.
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1 Introduction

In 1985, Oubina J. A. [1] introduced a new class of almost Contact manifold namely trans-Sasakian manifold. Many geometers like [2], [3], [4], [5], [6], [7], [8], [9] have studied this manifold and obtained many interesting results. The notion of quarter-symmetric connection generalizes the semi-symmetric connection. In 1924, Friedman A. and Schouten J. A. [10], [11] introduced the notion of semi-symmetric linear connection on a differentiable manifold. In 1975, Golab S. [12] defined and studied

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quarter-symmetric connection in a differentiable manifold with affine connection. Rastogi S. C. [13], [14] continued the systematic study of quarter symmetric metric connection.

A linear connection $\widetilde{\nabla}$ in $(2n+1)$ dimensional differentiable manifold is said to be a quarter symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y], \tag{1.1}$$
$$= \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.2}$$

where η is a 1-form and ϕ is a tensor of type $(1,1)$. In particular, if $\phi X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection. And if quarter-symmetric metric connection $\widetilde{\nabla}$ satisfies the condition

$$(\widetilde{\nabla}_X g)(Y,Z) = 0,$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on the manifold M, then $\widetilde{\nabla}$ is said to be a quarter symmetric metric connection.

In 1980, Mishra R. S. and Pandey S. N. [15] studied quarter symmetric metric connection in a Riemannian, Kaehlerian and Sasakian manifolds. In 1982, Yano K. and Imai T. [16] studied quarter symmetric metric connection in Hermition and Kaehlerian manifolds. In 1991, Mukhopadhyay S. et al. [17] studied quarter symmetric metric connection on a Riemannian manifold with an almost complex structure ϕ. Quarter symmetric metric connection was also studied by Singh R. N. [18]. Biswas and De U. C. [19], De U. C. and Mondal A. K. [20] studied Quarter symmetric metric connection on SP-Sasakian manifold and Sasakian manifolds. Pradeep Kumar K. T., Bagewadi C. S. and Venkatesha [21,22] and Prakasha D. G. [23] studied Quarter symmetric metric connection on K-contact manifolds and Kenmotsu manifolds. Shyamal Kumar Hui [24] studied ϕ-pseudo symmetric Kenmotsu manifolds with respect to quarter symmetric metric connection. Also Patra C. and Bhatt-acharya A. [25] studied trans-Sasakian manifold admitting quarter symmetric non-metric connection.

The notion of local symmetric of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetric, Takahashi T. [26] introduced the notion of locally ϕ-symmetry on a Sasakian manifold. In the context of Contact geometry the notion of ϕ-symmetric is introduced and studied by Boeckx E. Buecken P. and Vanhecke L. [27] with several examples. Motivated by these ideas, in the present paper we like to study quarter symmetric metric connection on a trans-Sasakian manifold. The present paper is organized as follows.

In section 2, we recall some preliminary results. In section 3, we give the relation between the Levi-Civita connection and the quarter symmetric metric connection on three dimensional trans-Sasakian manifold. Section 4 deals with locally ϕ-symmetric trans-Sasakian manifold with respect to quarter symmetric metric connection. In the next section we study ϕ-symmetric trans-Sasakian manifold with respect to quarter symmetric metric connection. Finally in section 6, we study locally concircular ϕ-symmetric trans-Sasakian manifold with respect to quarter symmetric metric connection.

2 Preliminaries

An $(2n+1)$ dimensional Riemannian manifold (M, g) is called an almost Contact manifold [28] if there exists on M, a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\phi X) = 0, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \tag{2.1}$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \tag{2.2}$$
$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.3}$$

for any vector fields X, Y on M.

An almost Contact metric structure (ϕ, ξ, η, g) on M is called a trans-Sasakian structure [1], if $(M \times R, J, G)$ belongs to the class W_4 [29], where J is the almost complex structure on $M \times R$ defined by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$ for any vector field X on M and smooth function f on $M \times R$. This may be expressed by the condition [30],

$$(\nabla_X \phi)(Y) = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{2.4}$$

for some smooth functions α and β on M, and we say that the trans-Sasakian structure is of type (α, β).
From (2.4), we have

$$(\nabla_X \xi) = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.5}$$

$$(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.6}$$

In a 3-dimensional trans-Sasakian manifold, from (2.4), (2.5) and (2.6), we can derive [31]

$$\begin{aligned} R(X,Y)Z &= [\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2)](g(Y,Z)X - g(X,Z)Y) \quad (2.7)\\ &- g(Y,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\xi \\ &- \eta(X)(\phi grad\alpha - grad\beta) + (X\beta + (\phi X)\alpha)\xi] \\ &+ g(X,Z)[(\frac{r}{2} + 2\xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\xi \\ &- \eta(Y)(\phi grad\alpha - grad\beta) + (Y\beta + (\phi Y)\alpha)\xi], \\ R(X,Y)\xi &= (\alpha^2 - \beta^2)\{\eta(Y)X - \eta(X)Y\} \quad (2.8)\\ &+ 2\alpha\beta\{\eta(Y)\phi X - \eta(X)\phi Y\} \\ &+ (Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y, \\ S(X,\xi) &= \left(2(\alpha^2 - \beta^2) - \xi\beta\right)\eta(X) - (\phi X)\alpha - X\beta. \quad (2.9) \end{aligned}$$

When $\phi(grad\alpha) = grad\beta$, (2.9) reduces to

$$S(X,\xi) = 2(\alpha^2 - \beta^2)\eta(X). \tag{2.10}$$

Definition 2.1. A Riemannian manifold M is said to be locally symmetric if $\nabla R = 0$, where R is the Riemannian curvature tensor of M and ∇ is Levi-Civita connection of M.

3 Curvature Tensor of a Three Dimensional Trans-Sasakian Manifold with Respect to Quarter Symmetric Metric Connection

Let $\widetilde{\nabla}$ be the linear connection and ∇ be the Riemannian connection of an almost Contact metric manifold such that

$$\widetilde{\nabla}_X Y = \nabla_X Y + H(X,Y), \tag{3.1}$$

where H is the tensor field of type $(1,1)$. For $\widetilde{\nabla}$ to be quarter-symmetric metric connection on M, we have [12]

$$H(X,Y) = \frac{1}{2}[T(X,Y) + T'(X,Y) + T'(Y,X)], \tag{3.2}$$

$$g(T'(X,Y), Z) = g(T'(Z,X), Y). \tag{3.3}$$

In view of equation (1.2) and (3.3), we obtain

$$T'(X,Y) = g(\phi Y, X)\xi - \eta(X)\phi Y. \tag{3.4}$$

Using (1.2) and (3.4) in (3.2), we get

$$H(X,Y) = -\eta(X)\phi Y.$$

Hence a quarter symmetric metric connection in a trans-Sasakian manifold is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{3.5}$$

This is the relation between quarter-symmetric metric connection and the Levi-Civita connection.

The curvature tensor $\widetilde{R}$ of M with respect to quarter symmetric metric connection $\widetilde{\nabla}$ is defined by

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z.$$

In view of (3.5), above equation takes the form

$$\begin{aligned} \widetilde{R}(X,Y)Z &= R(X,Y)Z - 2\alpha g(\phi Y, X)\phi Z \\ &- [\alpha g(X,Z) + \beta g(\phi X, Z)]\eta(Y)\xi + [\alpha g(Y,Z) + \beta g(\phi Y, Z)]\eta(X)\xi \\ &+ (\alpha X + \beta\phi X)\eta(Y)\eta(Z) - (\alpha Y + \beta \phi Y)\eta(X)\eta(Z), \end{aligned} \tag{3.6}$$

where $\widetilde{R}$ and R are the Riemannian curvature tensor with respect to $\widetilde{\nabla}$ and ∇ respectively. From equation (3.6) it follows that

$$\widetilde{S}(Y,Z) = S(Y,Z) - \alpha g(Y,Z) + \beta g(\phi Y, Z) + 3\alpha\eta(Y)\eta(Z), \tag{3.7}$$

where $\widetilde{S}$ and S are the Ricci tensor of the connections $\widetilde{\nabla}$ and ∇ respectively.
Contracting (3.7), we get

$$\widetilde{r} = r, \tag{3.8}$$

where $\widetilde{r}$ and r are the scalar curvature of the connections $\widetilde{\nabla}$ and ∇ respectively.

4 Locally ϕ-symmetric Trans-Sasakian Manifold with Respect to Quarter Symmetric Metric Connection

Definition 4.1. A trans-Sasakian manifold M is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{4.1}$$

for any arbitrary vector fields X, Y, Z, W orthogonal to ξ.

Analogous to this definition, we define a locally ϕ-symmetric trans-Sasakian manifold with respect to the quarter-symmetric metric connection by

$$\phi^2((\widetilde{\nabla}_W \widetilde{R})(X,Y)Z) = 0, \tag{4.2}$$

for all vector fields X, Y, Z and W orthogonal to ξ.
Using (3.5), we can write

$$(\widetilde{\nabla}_W \widetilde{R})(X,Y)Z = (\nabla_W \widetilde{R})(X,Y)Z) - \eta(W)\phi\widetilde{R}(X,Y)Z. \tag{4.3}$$

Now differentiating (3.6) with respect to W and using (2.4) and (2.6), we obtain

$$\begin{aligned}(\nabla_W\widetilde{R})(X,Y)Z &= (\nabla_W R)(X,Y)Z \\ &- 2\alpha g(\phi Y,X)[\alpha(g(W,Z)\xi-\eta(Z)W)+\beta(g(\phi W,Z)\xi \\ &- \eta(Z)\phi W)]-[\alpha g(X,Z)+\beta g(\phi X,Z)][\{-\alpha g(\phi W,Y) \\ &+ \beta g(\phi W,\phi Y)\}\xi+\eta(Y)\{-\alpha\phi W+\beta(W-\eta(W)\xi)\}] \\ &+ [\alpha g(Y,Z)+\beta g(\phi Y,Z)][\{-\alpha g(\phi W,X)+\beta g(\phi W,\phi X)\}\xi \\ &+ \eta(X)\{-\alpha\phi W+\beta(W-\eta(W)\xi)\}] \\ &+ (\alpha X+\beta\phi X)[\{-\alpha g(\phi W,Y)+\beta g(\phi W,\phi Y)\}\eta(Z) \\ &+ \eta(Y)\{-\alpha g(\phi W,Z)+\beta g(\phi W,\phi Z)\}]+\beta[\alpha(g(W,X)\xi \\ &- \eta(X)W)+\beta(g(\phi W,X)\xi-\eta(X)\phi W)]\eta(Y)\eta(Z) \\ &- (\alpha Y+\beta\phi Y)[\{-\alpha g(\phi W,X)+\beta g(\phi W,\phi X)\}\eta(Z) \\ &+ \eta(X)\{-\alpha g(\phi W,Z)+\beta g(\phi W,\phi Z)\}]-\beta[\alpha(g(W,Y)\xi \\ &- \eta(Y)W)+\beta(g(\phi W,Y)\xi-\eta(Y)\phi W)]\eta(X)\eta(Z).\end{aligned} \tag{4.4}$$

Using (2.1) and (4.4) in (4.3) and applying ϕ^2, we get

$$\begin{aligned}\phi^2((\widetilde{\nabla}_W\widetilde{R})(X,Y)Z) &= \phi^2((\nabla_W R)(X,Y)Z) \\ &+ 2\alpha g(\phi Y,X)[\alpha\phi^2 W+\beta\phi^2(\phi W)]\eta(Z) \\ &- \eta(Y)[\alpha g(X,Z)+\beta g(\phi X,Z)][-\alpha\phi^2(\phi W)+\beta\phi^2 W] \\ &+ \eta(X)[\alpha g(Y,Z)+\beta g(\phi Y,Z)][-\alpha\phi^2(\phi W)+\beta\phi^2 W] \\ &+ (\alpha\phi^2 X+\beta\phi^2(\phi X))[\{-\alpha g(\phi W,Y)+\beta g(\phi W,\phi Y)\}\eta(Z) \\ &+ \eta(Y)\{-\alpha g(\phi W,Z)+\beta g(\phi W,\phi Z)\}] \\ &- (\alpha\phi^2 Y+\beta\phi^2(\phi Y))[\{-\alpha g(\phi W,X)+\beta g(\phi W,\phi X)\}\eta(Z) \\ &+ \eta(X)\{-\alpha g(\phi W,Z)+\beta g(\phi W,\phi Z)\}] \\ &- \eta(W)\phi^2(\phi(\widetilde{R}(X,Y)Z)).\end{aligned} \tag{4.5}$$

If we consider X, Y, Z and W orthogonal to ξ then (4.5) reduces to

$$\phi^2((\widetilde{\nabla}_W\widetilde{R})(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z). \tag{4.6}$$

Hence we can state:

Theorem 4.1. *In a three dimensional trans-Sasakian manifold, the locally ϕ-symmetric remains invariant under quarter-symmetric metric connection $\widetilde{\nabla}$ and Levi-Civita connection ∇.*

5 ϕ-Symmetric Trans-Sasakian Manifold with Respect to the Quarter-Symmetric Metric Connection

Definition 5.1. A trans-Sasakian manifold M is said to be ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{5.1}$$

for any arbitrary vector fields X, Y, Z and W on M.

Similarly, a trans-Sasakian manifold M is said to be ϕ-symmetric with respect to the quarter-symmetric metric connection if

$$\phi^2((\widetilde{\nabla}_W\widetilde{R})(X,Y)Z) = 0, \tag{5.2}$$

for arbitrary vector fields X, Y, Z and W.
Now by virtue of (2.1), (5.2) gives

$$-(\widetilde{\nabla}_W \widetilde{R})(X,Y)Z + \eta((\widetilde{\nabla}_W \widetilde{R})(X,Y)Z)\xi = 0, \tag{5.3}$$

from which it follows that

$$-g((\widetilde{\nabla}_W \widetilde{R})(X,Y)Z,U) + \eta((\widetilde{\nabla}_W \widetilde{R})(X,Y)Z)\eta(U) = 0. \tag{5.4}$$

Let $\{e_i\}, i = 1,2,3$ be an orthonormal basis of the tangent space at any point of the space form. Then putting $X = U = e_i$, in (5.4) and taking summation over i, $1 \le i \le 3$, we get

$$-(\widetilde{\nabla}_W \widetilde{S})(Y,Z) + \sum \eta((\widetilde{\nabla}_W \widetilde{R})(e_i,Y)Z)\eta(e_i) = 0. \tag{5.5}$$

The second term of (5.5) by putting $Z = \xi$ takes the form

$$\eta((\widetilde{\nabla}_W \widetilde{R})(e_i,Y)\xi)\eta(e_i) = g((\widetilde{\nabla}_W \widetilde{R})(e_i,Y)\xi,\xi)g(e_i,\xi). \tag{5.6}$$

By using (3.5) and (4.3), we get

$$g((\widetilde{\nabla}_W \widetilde{R})(e_i,Y)\xi,\xi) = g((\nabla_W \widetilde{R})(e_i,Y)\xi,\xi) - \eta(W)\eta(\phi\widetilde{R}(e_i,Y)\xi). \tag{5.7}$$

On simplification we obtain from (5.7) that

$$g((\widetilde{\nabla}_W \widetilde{R})(e_i,Y)\xi,\xi) = g((\nabla_W R)(e_i,Y)\xi,\xi). \tag{5.8}$$

In a trans-Sasakian manifold M, we have $g((\nabla_W R)(e_i,Y)\xi,\xi) = 0$ and so from (5.8), we have

$$g((\widetilde{\nabla}_W \widetilde{R})(e_i,Y)\xi,\xi) = 0. \tag{5.9}$$

By replacing Z by ξ in (5.5) and using (5.9), we get

$$(\widetilde{\nabla}_W \widetilde{S})(Y,\xi) = 0. \tag{5.10}$$

We know that

$$(\widetilde{\nabla}_W \widetilde{S})(Y,\xi) = \widetilde{\nabla}_W \widetilde{S}(Y,\xi) - \widetilde{S}(\widetilde{\nabla}_W Y,\xi) - \widetilde{S}(Y,\widetilde{\nabla}_W \xi). \tag{5.11}$$

By making use of (2.4), (2.9), (3.5) and (3.7) in (5.11), we get

$$\begin{aligned} \beta S(Y,W) - \alpha S(Y,\phi W) &= 2(\alpha^2 - \beta^2 + \alpha)[-\alpha g(Y,\phi W) + \beta g(\phi Y,\phi W)] \\ &- (\alpha^2 - \beta^2)g(Y,\phi W) + 2\alpha\beta g(\phi Y,\phi W) \\ &+ 2\beta(\alpha^2 - \beta^2)\eta(W)\eta(Y). \end{aligned} \tag{5.12}$$

Replacing Y by ϕY and W by ϕW, we get

$$\begin{aligned} &\beta[S(Y,W) - 2(\alpha^2 - \beta^2 + \alpha)g(Y,W) + 2n\alpha\eta(Y)\eta(W) + \beta g(W,\phi Y) - \alpha g(\phi Y,\phi W)] \\ + \;&\alpha[S(\phi Y,W) - 2(\alpha^2 - \beta^2 + \alpha)g(\phi Y,W) - \beta g(\phi W,\phi Y) - \alpha g(\phi Y,W)] = 0. \end{aligned}$$

This implies

$$\begin{aligned} &S(Y,W) - 2(\alpha^2 - \beta^2 + \alpha)g(Y,W) + 2\alpha\eta(Y)\eta(W) + \beta g(W,\phi Y) \\ - \;&\alpha g(\phi Y,\phi W) = 0 \end{aligned} \tag{5.13}$$

$$and$$

$$S(\phi Y,W) - 2(\alpha^2 - \beta^2 + \alpha)g(\phi Y,W) - \beta g(\phi W,\phi Y) - \alpha g(\phi Y,W) = 0. \tag{5.14}$$

Contracting (5.13) and (5.14), we get

$$r = 6(\alpha^2 - \beta^2 + \alpha). \tag{5.15}$$

Therefore we can state,

Theorem 5.1. *Let M be a three dimensional ϕ-symmetric trans-Sasakian manifold with respect to quarter symmetric metric connection $\widetilde{\nabla}$. Then the manifold has a scalar curvature r with respect to Levi-Civita connection ∇ of M given by (5.15).*

Case I: Considering $\alpha = 1$ and $\beta = 0$ in (5.15) we get $r = 12$ which is proved by Pradeep Kumar K. T., Bagewadi C. S. and Venkatesha in [21] for a Sasakian manifold.

Case II: And considering $\alpha = 0$ and $\beta = 1$ in (5.15) we get $r = -6$ which is proved by Prakasha D. G. in [23] for a Kenmotsu manifold.

6 Locally Concircular ϕ-symmetric Trans-Sasakian Manifold with Respect to Quarter Symmetric Metric Connection

Definition 6.1. A trans-Sasakian manifold M is said to be locally concircular ϕ-symmetric if

$$\phi^2((\nabla_W \widetilde{C})(X,Y)Z) = 0, \tag{6.1}$$

for any arbitrary vector fields X, Y, Z, W orthogonal to ξ, where $\widetilde{C}$ is the concircular curvature tensor given by [32].

$$\widetilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{6}[g(Y,Z)X - g(X,Z)Y]. \tag{6.2}$$

A trans-Sasakian manifold M is said to be locally concircular ϕ-symmetric with respect to the quarter-symmetric metric connection if

$$\phi^2((\widetilde{\nabla}_W \widetilde{\widetilde{C}})(X,Y)Z) \quad = \quad 0. \tag{6.3}$$

for all vector field X, Y, Z and W orthogonal to ξ, where $\widetilde{\widetilde{C}}$ is the concircular curvature tensor with respect to quarter symmetric metric connection given by

$$\widetilde{\widetilde{C}}(X,Y)Z = \widetilde{R}(X,Y)Z - \frac{\widetilde{r}}{6}[g(Y,Z)X - g(X,Z)Y], \tag{6.4}$$

where $\widetilde{R}$ and $\widetilde{r}$ are the Riemannian curvature tensor and the scalar curvature with respect to quarter symmetric metric connection.
Using (3.5), we can write

$$(\widetilde{\nabla}_W \widetilde{\widetilde{C}})(X,Y)Z = (\nabla_W \widetilde{\widetilde{C}})(X,Y)Z) - \eta(W)\phi\widetilde{\widetilde{C}}(X,Y)Z. \tag{6.5}$$

Now differentiating (6.4) covariantly with respect to W, we obtain

$$(\widetilde{\nabla}_W \widetilde{\widetilde{C}})(X,Y)Z = (\nabla_W \widetilde{R})(X,Y)Z - \frac{\nabla_W \widetilde{r}}{6}[g(Y,Z)X - g(X,Z)Y]. \tag{6.6}$$

Using (3.8) and (4.4) in (6.6), we have

$$\begin{aligned}(\nabla_W\widetilde{\widetilde{C}})(X,Y)Z &= (\nabla_W R)(X,Y)Z - 2\alpha g(\phi Y,X)[\alpha(g(W,Z)\xi \\ &- \eta(Z)W) + \beta(g(\phi W,Z)\xi - \eta(Z)\phi W)] \\ &- [\alpha g(X,Z) + \beta g(\phi X,Z)][\{-\alpha g(\phi W,Y) \\ &+ \beta g(\phi W,\phi Y)\}\xi + \eta(Y)\{-\alpha\phi W + \beta(W - \eta(W)\xi)\}] \\ &+ [\alpha g(Y,Z) + \beta g(\phi Y,Z)][\{-\alpha g(\phi W,X) \\ &+ \beta g(\phi W,\phi X)\}\xi + \eta(X)\{-\alpha\phi W + \beta(W - \eta(W)\xi)\}] \\ &+ (\alpha X + \beta\phi X)[\{-\alpha g(\phi W,Y) + \beta g(\phi W,\phi Y)\}\eta(Z) \\ &+ \eta(Y)\{-\alpha g(\phi W,Z) + \beta g(\phi W,\phi Z)\}] + \beta[\alpha(g(W,X)\xi \\ &- \eta(X)W) + \beta(g(\phi W,X)\xi - \eta(X)\phi W)]\eta(Y)\eta(Z) \\ &- (\alpha Y + \beta\phi Y)[\{-\alpha g(\phi W,X) + \beta g(\phi W,\phi X)\}\eta(Z) \\ &+ \eta(X)\{-\alpha g(\phi W,Z) + \beta g(\phi W,\phi Z)\}] - \beta[\alpha(g(W,Y)\xi \\ &- \eta(Y)W) + \beta(g(\phi W,Y)\xi - \eta(Y)\phi W)]\eta(X)\eta(Z) \\ &- \frac{\nabla_W\widetilde{r}}{6}[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{6.7}$$

Again using (6.2), the equation (6.7) reduces to

$$\begin{aligned}(\nabla_W\widetilde{\widetilde{C}})(X,Y)Z &= (\nabla_W C)(X,Y)Z - 2\alpha g(\phi Y,X)[\alpha(g(W,Z)\xi \\ &- \eta(Z)W) + \beta(g(\phi W,Z)\xi - \eta(Z)\phi W)] \\ &- [\alpha g(X,Z) + \beta g(\phi X,Z)][\{-\alpha g(\phi W,Y) \\ &+ \beta g(\phi W,\phi Y)\}\xi + \eta(Y)\{-\alpha\phi W + \beta(W - \eta(W)\xi)\}] \\ &+ [\alpha g(Y,Z) + \beta g(\phi Y,Z)][\{-\alpha g(\phi W,X) \\ &+ \beta g(\phi W,\phi X)\}\xi + \eta(X)\{-\alpha\phi W + \beta(W - \eta(W)\xi)\}] \\ &+ (\alpha X + \beta\phi X)[\{-\alpha g(\phi W,Y) + \beta g(\phi W,\phi Y)\}\eta(Z) \\ &+ \eta(Y)\{-\alpha g(\phi W,Z) + \beta g(\phi W,\phi Z)\}] + \beta[\alpha(g(W,X)\xi \\ &- \eta(X)W) + \beta(g(\phi W,X)\xi - \eta(X)\phi W)]\eta(Y)\eta(Z) \\ &- (\alpha Y + \beta\phi Y)[\{-\alpha g(\phi W,X) + \beta g(\phi W,\phi X)\}\eta(Z) \\ &+ \eta(X)\{-\alpha g(\phi W,Z) + \beta g(\phi W,\phi Z)\}] - \beta[\alpha(g(W,Y)\xi \\ &- \eta(Y)W) + \beta(g(\phi W,Y)\xi - \eta(Y)\phi W)]\eta(X)\eta(Z).\end{aligned} \tag{6.8}$$

Using (2.1) and (6.8) in (6.5) and applying ϕ^2, we get

$$\begin{aligned}\phi^2(\widetilde{\nabla}_W\widetilde{\widetilde{C}})(X,Y)Z &= \phi^2((\nabla_W\widetilde{C})(X,Y)Z) + 2\alpha g(\phi Y,X)[\alpha\phi^2 W + \beta\phi^2(\phi W)]\eta(Z) \\ &- \eta(Y)[\alpha g(X,Z) + \beta g(\phi X,Z)][-\alpha\phi^2(\phi W) + \beta\phi^2 W] \\ &+ \eta(X)[\alpha g(Y,Z) + \beta g(\phi Y,Z)][-\alpha\phi^2(\phi W) + \beta\phi^2 W] \\ &+ (\alpha\phi^2 X + \beta\phi^2(\phi X))[\{-\alpha g(\phi W,Y) + \beta g(\phi W,\phi Y)\}\eta(Z) \\ &+ \eta(Y)\{-\alpha g(\phi W,Z) + \beta g(\phi W,\phi Z)\}] \\ &- (\alpha\phi^2 Y + \beta\phi^2(\phi Y))[\{-\alpha g(\phi W,X) + \beta g(\phi W,\phi X)\}\eta(Z) \\ &+ \eta(X)\{-\alpha g(\phi W,Z) + \beta g(\phi W,\phi Z)\}] \\ &- \eta(W)\phi^2(\phi(\widetilde{\widetilde{C}}(X,Y)Z)).\end{aligned} \tag{6.9}$$

If we consider X, Y, Z and W orthogonal to ξ, then (6.9) reduces to

$$\phi^2(\widetilde{\nabla}_W\widetilde{\widetilde{C}})(X,Y)Z = \phi^2((\nabla_W\widetilde{C})(X,Y)Z). \tag{6.10}$$

Hence we state:

Theorem 6.1. *In a three dimensional trans-Sasakian manifold, locally concircular ϕ-symmetric remains invariant under quarter symmetric metric $\widetilde{\nabla}$ and Levi-Civita connection ∇.*

In view of (2.1) and (6.7), we have from (6.5) that

$$\begin{aligned}
\phi^2(\widetilde{\nabla}_W\widetilde{\widetilde{C}})(X,Y)Z &= \phi^2((\nabla_W R)(X,Y)Z) + 2\alpha g(\phi Y, X)[\alpha\phi^2 W + \beta\phi^2(\phi W)]\eta(Z) \\
&- \eta(Y)[\alpha g(X,Z) + \beta g(\phi X, Z)][-\alpha\phi^2(\phi W) + \beta\phi^2 W] \\
&+ \eta(X)[\alpha g(Y,Z) + \beta g(\phi Y, Z)][-\alpha\phi^2(\phi W) + \beta\phi^2 W] \\
&+ (\alpha\phi^2 X + \beta\phi^2(\phi X))[\{-\alpha g(\phi W, Y) + \beta g(\phi W, \phi Y)\}\eta(Z) \\
&+ \eta(Y)\{-\alpha g(\phi W, Z) + \beta g(\phi W, \phi Z)\}] \\
&- (\alpha\phi^2 Y + \beta\phi^2(\phi Y))[\{-\alpha g(\phi W, X) + \beta g(\phi W, \phi X)\}\eta(Z) \\
&+ \eta(X)\{-\alpha g(\phi W, Z) + \beta g(\phi W, \phi Z)\}] \\
&- \eta(W)\phi^2(\phi(\widetilde{\widetilde{C}}(X,Y)Z)) \\
&- \frac{\nabla_W\widetilde{r}}{6}[g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y].
\end{aligned} \tag{6.11}$$

If we consider X, Y, Z and W orthogonal to ξ, then (6.11) reduces to

$$\begin{aligned}
\phi^2(\widetilde{\nabla}_W\widetilde{\widetilde{C}})(X,Y)Z &= \phi^2((\nabla_W R)(X,Y)Z) \\
&- \frac{\nabla_W\widetilde{r}}{6}[g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y].
\end{aligned} \tag{6.12}$$

If r is constant, then $\nabla_W r = 0$. Therefore (6.12) yields

$$\phi^2(\widetilde{\nabla}_W\widetilde{\widetilde{C}})(X,Y)Z = \phi^2((\nabla_W R)(X,Y)Z). \tag{6.13}$$

Hence we state:

Theorem 6.2. *A three dimensional locally concircular ϕ-symmetric trans-Sasakian manifold admitting quarter symmetric metric connection is locally ϕ-symmetric if and only if the scalar curvature r is constant with respect to Levi-Civita connection.*

Competing Interests

The authors declare that no competing interests exist.

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