Journal of Advances in Mathematics and Computer Science

30(1): 1-11, 2019; Article no.JAMCS.45973

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

A New Projection Type Algorithm for Compressive Sensing

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Authors' contributions

The author BZ has proposed a new projection-type algorithm, has established its global convergence, and has accomplished the numerical results. Author HS has proposed the motivations of the manuscript. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2019/45973
Editor(s):
(1) Dr. Mangatiana A. Robdera, Professor, Department of Mathematics, University of Botswana, Botswana.
Reviewers:
(1) Hui Yang, Beijing University of Posts and Telecommunications (BUPT), China.
(2) Anand Nayyar, Duy Tan University, Vietnam.
Complete Peer review History: http://www.sciencedomain.org/review-history/27949

Received: 23 September 2018
Accepted: 15 December 2018
Published: 24 December 2018

Original Research Article

Abstract

Compressive sensing (CS) is to recover a sparse signal from an undetermined linear system, which has received considerable interest, and some customized iterative methods for solving CS have been proposed in recent years. In this paper, we further consider an algorithm for solving the CS. To this end, a new projection-type algorithm (PTA) is proposed to solve CS based on a new formulation of the problem, which needs only one projection onto the nonnegative quadrant and only one value of the mapping per iteration. Global convergence results of the new algorithm is established. Furthermore, we illustrate the efficiency of given algorithm through some numerical examples on sparse signal recovery.

Keywords: Compressive sensing; projection-type algorithm; global convergence.

2010 Mathematics Subject Classification: 65H10, 90C33.

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1 Introduction

Compressive sensing (CS) is to recover a sparse signal $\bar{x} \in R^n$ from an undetermined linear system $y = A\bar{x}$, where $A \in R^{m\times n}$ $(m \ll n)$ is the sensing matrix. A fundamental decoding model in CS is the following unconstrained basis pursuit denoising (abbreviated as BPD) problem, which can be mathematically depicted as

$$\min_{x\in R^n} \frac{1}{2}\|Ax - y\|_2^2 + \rho\|x\|_1, \tag{1.1}$$

where $\rho > 0$ is a parameter and $\|x\|_1$ is the ℓ_1-norm of the vector x. Throughout this paper, we assume that the solution set of 1.1 is nonempty.

Obviously, the function $\|x\|_1$ is convex although it is not differential, 1.1 is a convex optimization problem, and there are some standard methods such as the smooth Newton-type methods or interior-point methods to solve it [1, 2, 3, 4]. However, these solvers are not tailored for large-scale cases of CS and they become inefficient as n increases. In recent years, some customized iterative methods for 1.1 have been proposed [5, 6, 7, 8, 9, 10, 11]. In the following, we briefly summarize some influential methods for 1.1. Landi [4] propose to solve 1.1 by an efficient modified Newton projection method only requiring matrix-vector operations. Li, Sun and Toh [5] develop an algorithm for solving large-scale convex composite optimization models with an emphasis on the ℓ_1-regularized least squares regression (lasso) problems. In [12] a spectral gradient method is applied to solve problem 1.1 without requiring Jacobian matrix information. In [13], based on Bregman iterative regularization, the authors proposed some efficient methods for solving the compressed sensing. Hale et al. [14] presented a framework for solving the large-scale ℓ_1-regularized convex minimization problem based on operator-splitting and continuation. Yin et al. [15] presented an iterative method for ℓ_{1-2} minimization based on the difference of convex functions algorithm. Lou and Yan [16] also give a method via the proximal operator for the $\ell_1 - \ell_2$ minimization. For $\ell_2 - \ell_p$ minimization problem, based on fixed-point iterations, a projection-based algorithm was presented by Borges et al. [17]. For the same problem, Chen et al. [18] developed a lower bound to classify zero and nonzero entries in every local solution, and also develop error bounds. Based on these results, the authors proposed a hybrid OMP-SG method for solving it. In addition, BPD problem is obvious a special case of the famous separable convex programming, some the numerical methods which can solve the separable convex programming are applicable to the above BPD problem (e.g., [19, 20, 21, 22, 23, 24, 25]). In this paper, we shall propose a new projection-type algorithm for the problem BPD with a closed form, whose iterative scheme doesn't also need to perform a backtracking line search at each iteration.

The rest of this paper is organized as follows. In Section 2, we give some equivalent reformulations of the problem BPD. In Section 3, some related properties are given, which are the basis of our analysis. In Section 4, we propose a new projection-type algorithm without the backtracking line search to find a suitable step size, which needs only one projection onto the nonnegative quadrant and only one value of the mapping per iteration. We show that the new PTA is global convergence in detail. In Section 5, some numerical experiments on compressive sensing are given to show the efficiency of the proposed method. Finally, some conclusions and remarks are presented in Section 6.

To end this section, some notations used in this paper are in order. We use R^n_+ to denote the nonnegative quadrant in R^n, and the x_+ denotes the orthogonal projection of vector $x \in R^n$ onto R^n_+, that is, $(x_+)_i := \max\{x_i, 0\},\ 1 \le i \le n$; the norm $\|\cdot\|$ and $\|\cdot\|_1$ denote the Euclidean 2-norm and 1-norm, respectively. For $x, y \in R^n$, use $(x; y)$ to denote the column vector $(x^\top, y^\top)^\top$.

2 Equivalent Reformulations of the Problem BPD

In this section, we will establish some smooth equivalence transformations of the problem BPD. Firstly, we define two variables μ_i and ν_i $(i = 1, 2, \cdots, n)$ as [12]:

$$\mu_i + \nu_i = |x_i|, \mu_i - \nu_i = x_i, i = 1, 2, \cdots, n.$$

Thus, we can reformulate BPD into

$$\begin{array}{ll} \min_{(\mu;\nu)\in R^{2n}} & \frac{1}{2}\|(A, -A)(\mu; \nu) - y\|_2^2 + \rho(e^\top, e^\top)(\mu; \nu) \\ \text{s.t.} & (\mu; \nu) \geq 0, \end{array} \tag{2.1}$$

where $e \in R^n$ denotes the vector composed by elements 1, i.e., $e = (1, 1, \cdots, 1)^\top$.

To make the description more concise, letting $\omega = (\mu; \nu)$. we have the following equivalent formulation of (2.1)

$$\begin{array}{ll} \min & f(\omega) = \frac{1}{2}(\omega^\top M\omega - 2p^\top \omega + y^\top y) \\ \text{s.t.} & \omega \in R^{2n}_+, \end{array} \tag{2.2}$$

where $M = \begin{pmatrix} A^\top A, & -A^\top A \\ -A^\top A, & A^\top A \end{pmatrix}, p = \begin{pmatrix} A^\top \\ -A^\top \end{pmatrix} y - \rho \begin{pmatrix} e \\ e \end{pmatrix}$.

Obviously, the problem (2.2) is a convex optimization problem, then the stationary set of 2.2 coincides with its solution set which also coincides with the solution set of the following the problem: find $\omega^* \in R^{2n}_+$ such that

$$(\omega - \omega^*)^\top (M\omega^* - p) \geq 0, \quad \forall \omega \in R^{2n}_+. \tag{2.3}$$

In the meantime, the system (2.3) can be further written as the following linear complementarity problem: find $\omega^* \in R^{2n}_+$ such that

$$\omega^* \geq 0, \quad M\omega^* - p \geq 0, \quad (\omega^*)^\top (M\omega^* - p) = 0. \tag{2.4}$$

The solution set of 2.4 is nonempty under the nonempty assumption of the solution of 1.1, and is denoted by Ω^*. 2.4 is also an equivalent reformulation of the problem BPD.

3 Preliminaries

In this section, based on the equivalent reformulations 2.3) and (2.4 in Section 2, we will give the definition of projection operator and some related properties [26, 27], which are the basis of our analysis.

For a nonempty closed convex set $K \subset R^n$ and vector $x \in R^n$, the orthogonal projection of x onto K, i.e., $\arg\min\{\|y - x\| | y \in K\}$, is denoted by $P_K(x)$.

Proposition 3.1. *Let K be a closed convex subset of R^n.For any $x, y \in R^n$ and $z \in K$, the following statements hold.*

$$\begin{array}{ll} (i) & \langle P_K(x) - x, z - P_K(x)\rangle \geq 0. \\ (ii) & |P_K(x) - z\|^2 \leq \|x - z\|^2 - \|P_K(x) - x\|^2. \end{array}$$

For the classical variational inequality problem(VIP) which is to find a point $x^* \in K$ such that

$$F(x^*)^\top (x - x^*) \geq 0, \forall x \in K,$$

where K is a closed convex set in R^n, we have the following conclusion.

Proposition 3.2. *(Minty, see, e.g., [28], Lemma 7.1.7) Assume that $F : K \to R^n$ is continuous and monotone mapping. Then x^* is a solution of VIP if and only if is a solution of the following problem: find $x \in K$ such that*

$$F(x)^\top (x - x^*) \geq 0, \quad \forall x \in K.$$

For (2.3),(2.4) and $\omega \in R^{2n}$, define the projection residue

$$R(\omega) := \omega - P_{R^{2n}_+}(\omega - \beta F(\omega)) = \min\{\omega, \beta F(\omega)\},$$

where $\beta > 0$ is a constant, $F(\omega) = M\omega - p$. The projection residue is intimately related to the solution of 2.3 and 2.4 as shown by the following well-known result, which is due to Noor [29].

Proposition 3.3. *ω^* is a solution of (2.3) if and only if $R(\omega^*) = 0$ with some $\beta > 0$.*

By Proposition 3.3, we have that solutions of (1.1) coincide with zeros of the following projected residual function:

$$r(\omega, z) := \|z - P_{R^{2n}_+}(\omega - \beta F(z))\| + \|\omega - z\|,$$

where $\beta > 0$ is a constant.

4 Algorithm and Global Convergence

In this section, we will propose a new projection-type algorithm (PTA) to solve BPD with a closed form, and prove global convergence of the new PTA in detail. Now, we formally state our algorithm.

Algorithm 4.1.

Step0. Select $\omega_0 = z_0 \in R^{2n}$, $\sigma > 1$, $\beta \in (0, \frac{(\sqrt{2}-1)\sigma}{(\sigma-1)\|M\|})$,and let $k := 0$.

Step1. For the current iterate points ω^k and z^k, compute

$$\omega_{k+1} = \{\omega_k - \beta F(z_k)\}_+. \tag{4.1}$$

If $r(\omega_k, z_k) = 0$, stop. Then $\omega_k = z_k$ is a solution of (2.3). Otherwise, go to Step 2.

Step2. compute $z_{k+1} \in R^{2n}_+$ such that $z_{k+1} = \frac{2\sigma-1}{\sigma}\omega_{k+1} - \frac{\sigma-1}{\sigma}\omega_k$. Go to Step 1 with $k \overset{\triangle}{=} k + 1$.

Remark 4.1. It is easy to see that this method needs only one projection onto the set R^{2n}_+ and only one value of $F(\omega)$ per iteration. Therefor, it makes algorithm 4.1 very attractive for cases when a computation of operator F is expensive.

To establish the convergence of the Algorithm 4.1, we first give the following two lemmas, which is a basis for further discussion.

Lemma 4.1. *Let $\{\omega_k\}$ and $\{z_k\}$ be two sequences generated by Algorithm 4.1 . Then*

$$2\beta F(z_{k-1})^\top (z_k - \omega_{k+1}) \quad \leq \frac{\sigma}{\sigma-1}\{\|\omega_{k+1} - \omega_k\|^2 - \|\omega_k - z_k\|^2 - \|\omega_{k+1} - z_k\|^2\}. \tag{4.2}$$

Proof. Applying Proposition 3.1 (i), (4.1) with $\omega_k = P_{R^{2n}_+}\{\omega_{k-1} - \beta F(z_{k-1})\}$, $\omega_{k+1} \in R^{2n}_+$, and $\sigma > 1$, one has

$$\sigma[\omega_k - (\omega_{k-1} - \beta F(z_{k-1})]^\top (\omega_k - \omega_{k+1}) \leq 0, \tag{4.3}$$

$$(\sigma - 1)[\omega_k - (\omega_{k-1} - \beta F(z_{k-1})]^\top (\omega_k - \omega_{k-1}) \leq 0, \tag{4.4}$$

Combining (4.3) with (4.4) yields

$$\begin{aligned} 0 \quad &\geq [\omega_k - (\omega_{k-1} - \beta F(z_{k-1})]^\top ((2\sigma - 1)\omega_k - \sigma\omega_{k+1} - (\sigma - 1)\omega_{k-1}) \\ &= \sigma[\omega_k - (\omega_{k-1} - \beta F(z_{k-1})]^\top (z_k - \omega_{k+1}) \\ &= \sigma[(\omega_k - \omega_{k-1}) + \beta F(z_{k-1})]^\top (z_k - \omega_{k+1}), \end{aligned} \tag{4.5}$$

where the first equality is by $\sigma z_k = (2\sigma - 1)\omega_k - (\sigma - 1)\omega_{k-1}$. From (4.5), we obtain

$$\begin{array}{ll} 2\beta F(z_{k-1})^\top (z_k - \omega_{k+1}) & \leq 2(\omega_k - \omega_{k-1})^\top (\omega_{k+1} - z_k) \\ & = \frac{2\sigma}{\sigma-1}(z_k - \omega_k)^\top (\omega_{k+1} - z_k) \\ & = \frac{\sigma}{\sigma-1}\{\|\omega_k - z_k\|^2 + \|\omega_{k+1} - z_k\|^2 + 2(z_k - \omega_k)^\top (\omega_{k+1} - z_k)\} \\ & \quad - \frac{\sigma}{\sigma-1}\{\|\omega_k - z_k\|^2 + \|\omega_{k+1} - z_k\|^2\} \\ & = \frac{\sigma}{\sigma-1}\{\|(\omega_{k+1} - z_k) + (z_k - \omega_k)\|^2 - \|\omega_k - z_k\|^2 - \|\omega_{k+1} - z_k\|^2\}, \\ & = \frac{\sigma}{\sigma-1}\{\|\omega_{k+1} - \omega_k\|^2 - \|\omega_k - z_k\|^2 - \|\omega_{k+1} - z_k\|^2\}, \end{array} \tag{4.6}$$

where the first equality is by $(\sigma - 1)(\omega_k - \omega_{k-1}) = \sigma(\omega_k - z_k)$. □

Lemma 4.2. *Let $\{\omega_k\}$ and $\{z_k\}$ be two sequences generated by Algorithm 4.1. Then*

$$\begin{array}{ll} 2\beta[F(z_k) - F(z_{k-1})]^\top (z_k - \omega_{k+1}) \leq & (1 + \sqrt{2})\beta\|M\|\|z_k - \omega_k\|^2 + \beta\|M\|\|\omega_k - z_{k-1}\|^2 \\ & +\sqrt{2}\beta\|M\|\|z_k - \omega_{k+1}\|^2. \end{array} \tag{4.7}$$

Proof. By a direct computation yields that

$$\begin{array}{ll} & 2\beta[F(z_k) - F(z_{k-1})]^\top (z_k - \omega_{k+1}) \\ \leq & 2\beta\|F(z_k) - F(z_{k-1})\|\|z_k - \omega_{k+1}\| \\ \leq & 2\beta\|M\|\|z_k - z_{k-1}\|\|z_k - \omega_{k+1}\| \\ \leq & \beta\|M\|(\frac{1}{\sqrt{2}}\|z_k - z_{k-1}\|^2 + \sqrt{2}\|z_k - \omega_{k+1}\|^2) \\ = & \frac{\beta\|M\|}{\sqrt{2}}\|z_k - \omega_k + \omega_k - z_{k-1}\|^2 + \sqrt{2}\beta\|M\|\|z_k - \omega_{k+1}\|^2 \\ = & \frac{\beta\|M\|}{\sqrt{2}}[\|z_k - \omega_k\|^2 + \|\omega_k - z_{k-1}\|^2 + 2(z_k - \omega_k)^\top (\omega_k - z_{k-1})] + \sqrt{2}\beta\|M\|\|z_k - \omega_{k+1}\|^2 \\ \leq & \frac{\beta\|M\|}{\sqrt{2}}[\|z_k - \omega_k\|^2 + \|\omega_k - z_{k-1}\|^2 + 2\|z_k - \omega_k\|\|\omega_k - z_{k-1}\|] + \sqrt{2}\beta\|M\|\|z_k - \omega_{k+1}\|^2 \\ \leq & \frac{\beta\|M\|}{\sqrt{2}}[\|z_k - \omega_k\|^2 + \|\omega_k - z_{k-1}\|^2 + (\sqrt{2}+1)\|z_k - \omega_k\|^2 + (\sqrt{2}-1)\|\omega_k - z_{k-1}\|] \\ & +\sqrt{2}\beta\|M\|\|z_k - \omega_{k+1}\|^2 \\ \leq & \frac{\beta\|M\|}{\sqrt{2}}((\sqrt{2}+2)\|z_k - \omega_k\|^2 + \sqrt{2}\|\omega_k - z_{k-1}\|^2) + \sqrt{2}\beta\|M\|\|z_k - \omega_{k+1}\|^2 \\ \leq & (1 + \sqrt{2})\beta\|M\|\|z_k - \omega_k\|^2 + \beta\|M\|\|\omega_k - z_{k-1}\|^2 + \sqrt{2}\beta\|M\|\|z_k - \omega_{k+1}\|^2. \end{array}$$

□

Lemma 4.3. *Let $\{\omega_k\}$ and $\{z_k\}$ be two sequences generated by Algorithm 4.1, and let $\omega^* \in \Omega^*$. Then*

$$\begin{array}{ll} \|\omega_{k+1} - \omega^*\|^2 \leq & \|\omega_k - \omega^*\|^2 - (\frac{\sigma}{\sigma-1} - (1 + \sqrt{2})\beta\|M\|)\|z_k - \omega_k\|^2 \\ & + \beta\|M\|\|\omega_k - z_{k-1}\|^2 - (\frac{\sigma}{\sigma-1} - \sqrt{2}\beta\|M\|)\|z_k - \omega_{k+1}\|^2 \\ & - 2\beta F(\omega^*)^\top (z_k - \omega^*). \end{array} \tag{4.8}$$

Proof. Applying Proposition 3.1 (*ii*) and (4.1), one has

$$\begin{aligned}
\|\omega_{k+1}-\omega^*\|^2 &= \|\{\omega_k-\beta F(z_k)\}_+-\omega^*\|^2\\
&\le \|\omega_k-\beta F(z_k)-\omega^*\|^2-\|\omega_k-\beta F(z_k)-\omega_{k+1}\|^2\\
&\le \|\omega_k-\omega^*\|^2+\beta^2\|F(z_k)\|^2-2\beta F(z_k)^\top(\omega_k-\omega^*)\\
&\quad -\|\omega_{k+1}-\omega_k\|^2-\beta^2\|F(z_k)\|^2-2\beta F(z_k)^\top(\omega_{k+1}-\omega_k)\\
&= \|\omega_k-\omega^*\|^2-\|\omega_{k+1}-\omega_k\|^2-2\beta F(z_k)^\top(\omega_{k+1}-\omega^*)\\
&\le \|\omega_k-\omega^*\|^2-\|\omega_{k+1}-\omega_k\|^2-2\beta F(z_k)^\top(\omega_{k+1}-\omega^*)\\
&\quad +2\beta(F(z_k)-F(\omega^*))^\top(z_k-\omega^*)\\
&= \|\omega_k-\omega^*\|^2-\|\omega_{k+1}-\omega_k\|^2-2\beta F(z_k)^\top(\omega_{k+1}-\omega^*)\\
&\quad +2\beta F(z_k)^\top(z_k-\omega^*)-2\beta F(\omega^*)^\top(z_k-\omega^*)\\
&= \|\omega_k-\omega^*\|^2-\|\omega_{k+1}-\omega_k\|^2+2\beta F(z_k)^\top(z_k-\omega_{k+1})-2\beta F(\omega^*)^\top(z_k-\omega^*)\\
&= \|\omega_k-\omega^*\|^2-\|\omega_{k+1}-\omega_k\|^2+2\beta(F(z_k)-F(z_{k-1}))^\top(z_k-\omega_{k+1})\\
&\quad +2\beta F(z_{k-1})^\top(z_k-\omega_{k+1})-2\beta F(\omega^*)^\top(z_k-\omega^*)\\
&\le \|\omega_k-\omega^*\|^2-\|\omega_{k+1}-\omega_k\|^2+(1+\sqrt{2})\beta\|M\|\|z_k-\omega_k\|^2+\beta\|M\|\|\omega_k-z_{k-1}\|^2\\
&\quad +\sqrt{2}\beta\|M\|\|z_k-\omega_{k+1}\|^2+\tfrac{\sigma}{\sigma-1}\{\|\omega_{k+1}-\omega_k\|^2-\|\omega_k-z_k\|^2-\|\omega_{k+1}-z_k\|^2\}\\
&\quad -2\beta F(\omega^*)^\top(z_k-\omega^*)\\
&\le \|\omega_k-\omega^*\|^2-(\tfrac{\sigma}{\sigma-1}-(1+\sqrt{2})\beta\|M\|)\|z_k-\omega_k\|^2+\beta\|M\|\|\omega_k-z_{k-1}\|^2\\
&\quad -(\tfrac{\sigma}{\sigma-1}-\sqrt{2}\beta\|M\|)\|z_k-\omega_{k+1}\|^2-2\beta F(\omega^*)^\top(z_k-\omega^*).
\end{aligned} \tag{4.9}$$

where the third inequality holds since the matrix M is positive definite, the fourth inequality is by (4.2)and (4.7), the last inequality follows from $\sigma > 0$. □

Now, we give the global convergence result of algorithm 4.1.

Theorem 4.4. *Suppose that the solution set of (1.1) is nonempty, and the sequence $\{\omega^k\}$ generated by Algorithm 4.1 is an infinite generates, Then, the sequence $\{\omega^k\}$ is bounded and globally converges to a solution of (2.3).*

Proof. Firstly, we prove that the sequence $\{\omega^k\}$ is bounded.
From (4.8), by a direct computation yields that

$$\begin{aligned}
\|\omega_{k+1}-\omega^*\|^2 &\le \|\omega_k-\omega^*\|^2-(\tfrac{\sigma}{\sigma-1}-\beta\|M\|(1+\sqrt{2}))\|z_k-\omega_k\|^2\\
&\quad +\beta\|M\|\|\omega_k-z_{k-1}\|^2-(\tfrac{\sigma}{\sigma-1}-\sqrt{2}\beta\|M\|)\|z_k-\omega_{k+1}\|^2\\
&\quad -2\beta F(\omega^*)^\top(z_k-\omega^*)\\
&\le \|\omega_k-\omega^*\|^2-(\tfrac{\sigma}{\sigma-1}-\beta\|M\|(1+\sqrt{2}))\|z_k-\omega_k\|^2\\
&\quad +\beta\|M\|\|\omega_k-z_{k-1}\|^2-(\tfrac{\sigma}{\sigma-1}-\sqrt{2}\beta\|M\|)\|z_k-\omega_{k+1}\|^2\\
&\quad -2\beta F(\omega^*)^\top(\tfrac{2\sigma-1}{\sigma}\omega_k-\tfrac{\sigma-1}{\sigma}\omega_{k-1}-\omega^*)+2\beta F(\omega^*)^\top(\omega_{k-1}-\omega^*)\\
&= \|\omega_k-\omega^*\|^2-(\tfrac{\sigma}{\sigma-1}-\beta\|M\|(1+\sqrt{2}))\|z_k-\omega_k\|^2\\
&\quad +\beta\|M\|\|\omega_k-z_{k-1}\|^2-(\tfrac{\sigma}{\sigma-1}-\sqrt{2}\beta\|M\|)\|z_k-\omega_{k+1}\|^2\\
&\quad -2\beta F(\omega^*)^\top[(\tfrac{2\sigma-1}{\sigma}\omega_k-\tfrac{2\sigma-1}{\sigma}\omega^*)-(\tfrac{\sigma-1}{\sigma}\omega_{k-1}-\tfrac{\sigma-1}{\sigma}\omega^*)]\\
&\quad +2\beta F(\omega^*)^\top(\omega_{k-1}-\omega^*)\\
&\le \|\omega_k-\omega^*\|^2-(\tfrac{\sigma}{\sigma-1}-\beta\|M\|(1+\sqrt{2}))\|z_k-\omega_k\|^2\\
&\quad +\beta\|M\|\|\omega_k-z_{k-1}\|^2-\beta\|M\|\|z_k-\omega_{k+1}\|^2\\
&\quad -\tfrac{2(2\sigma-1)}{\sigma}\beta F(\omega^*)^\top(\omega_k-\omega^*)+\tfrac{2(2\sigma-1)}{\sigma}\beta F(\omega^*)^\top(\omega_{k-1}-\omega^*).
\end{aligned} \tag{4.10}$$

where the second inequality follows from the fact $z_k = \frac{2\sigma-1}{\sigma}\omega_k - \frac{\sigma-1}{\sigma}\omega_{k-1}$, using $\frac{\sigma}{\sigma-1} - \sqrt{2}\beta\|M\| \geq \beta\|M\|$, we obtain that the third inequality holds. Using (4.10), one has

$$\begin{aligned} & \|\omega_{k+1} - \omega^*\|^2 + \beta\|M\|\|z_k - \omega_{k+1}\|^2 + \tfrac{2(2\sigma-1)}{\sigma}\beta F(\omega^*)^\top(\omega_k - \omega^*) \\ \leq \; & \|\omega_k - \omega^*\|^2 + \beta\|M\|\|\omega_k - z_{k-1}\|^2 + \tfrac{2(2\sigma-1)}{\sigma}\beta F(\omega^*)^\top(\omega_{k-1} - \omega^*) \\ & -(\tfrac{\sigma}{\sigma-1} - \beta\|M\|(1+\sqrt{2}))\|z_k - \omega_k\|^2. \end{aligned} \tag{4.11}$$

Letting

$$a_k = \|\omega_k - \omega^*\|^2 + \beta\|M\|\|\omega_k - z_{k-1}\|^2 + \frac{2(2\sigma-1)}{\sigma}\beta F(\omega^*)^\top(\omega_{k-1} - \omega^*),$$
$$b_k = (\frac{\sigma}{\sigma-1} - \beta\|M\|(1+\sqrt{2}))\|z_k - \omega_k\|^2.$$

Combining this with (4.11), we obtain $a_{k+1} \leq a_k - b_k$. Since $\frac{\sigma}{\sigma-1} - (1+\sqrt{2})\beta\|M\| > 0$ and $\sigma > 1$, so $\{a_n\}$ and $\{b_n\}$ be two non-negative real sequences. Then the nonnegative sequence $\{a_n\}$ is strictly decreasing, so it converges, we also have $\lim_{n\to\infty} b_n = 0$, i.e., $\lim_{k\to\infty}\|z_k - \omega_k\| = 0$. Combining this with $(\sigma-1)(\omega_k - \omega_{k-1}) = \sigma(\omega_k - z_k)$, we obtain

$$\lim_{k\to\infty}\|\omega_k - \omega_{k-1}\| = \frac{\sigma}{\sigma-1}\lim_{k\to\infty}\|z_k - \omega_k\| = 0. \tag{4.12}$$

Moreover, $\{a_k\}$ is bounded since it is convergent. By the definition of a_n, one has

$$\|\omega_k - \omega^*\|^2 \leq a_k,$$

we obtain the sequence $\{\|\omega_k - \omega^*\|\}$ is bounded, and the sequence $\{\omega^k\}$ is bounded. Thus, there exists a subsequence $\{\omega_{k_i}\}$ of $\{\omega_k\}$ with $\omega_{k_i} \to \hat{\omega}$ as $k_i \to \infty$. Combining this with $\lim_{k\to\infty}\|z_k - \omega_k\| = 0$, one has

$$\lim_{k_i\to\infty}\|z_{k_i} - \hat{\omega}\| \leq \lim_{k_i\to\infty}\|z_{k_i} - \omega_{k_i}\| + \lim_{k_i\to\infty}\|\omega_{k_i} - \hat{\omega}\| = 0.$$

In the following, we will show that $\hat{\omega} \in \Omega^*$.

From Proposition 3.1 (i), for any $\omega \geq 0$, we have

$$\begin{aligned} 0 \; & \leq [\omega_{k_i+1} - (\omega_{k_i} - \beta F(z_{k_i}))]^\top(\omega - \omega_{k_i+1}) \\ & = (\omega_{k_i+1} - \omega_{k_i})^\top(\omega - \omega_{k_i+1}) + \beta F(z_{k_i})^\top(\omega - \omega_{k_i+1}) \\ & = (\omega_{k_i+1} - \omega_{k_i})^\top(\omega - \omega_{k_i+1}) + \beta F(z_{k_i})^\top(\omega - z_{k_i}) + \beta F(z_{k_i})^\top(z_{k_i} - \omega_{k_i+1}) \\ & \leq (\omega_{k_i+1} - \omega_{k_i})^\top(\omega - \omega_{k_i+1}) + \beta F(\omega)^\top(\omega - z_{k_i}) + \beta F(z_{k_i})^\top(z_{k_i} - \omega_{k_i+1}). \end{aligned} \tag{4.13}$$

where the last inequality follows from the fact

$$(F(\omega) - F(z_{k_i}))^\top(\omega - z_{k_i}) = (\omega - z_{k_i})^\top M(\omega - z_{k_i}) \geq 0.$$

Applying the fact that $\lim_{k_i\to\infty}\|\omega_{k_i+1} - \omega_{k_i}\| = 0$ and $\lim_{k_i\to\infty}\|z_{k_i} - \omega_{k_i}\| = 0$, we conclude that

$$\lim_{k_i\to\infty}\|z_{k_i} - \omega_{k_i+1}\| \leq \lim_{k_i\to\infty}\|z_{k_i} - \omega_{k_i}\| + \lim_{k_i\to\infty}\|\omega_{k_i+1} - \omega_{k_i}\| = 0.$$

Combining this with $\lim_{k_i\to\infty} z_{k_i} = \hat{\omega}$, and taking the limit as $k_i \to \infty$ in (4.13), we have

$$F(\omega)^\top(\omega - \hat{\omega}) \geq 0, \forall \omega \geq 0.$$

Combining this with Proposition 3.2, we obtain $\hat{\omega} \in \Omega^*$.

Secondly, From (4.12), we deduce that

$$\lim_{k\to\infty}\|\omega_k - z_{k-1}\| \lim_{k\to\infty}\|\omega_k - \omega_{k-1}\| + \lim_{k\to\infty}\|\omega_{k-1} - z_{k-1}\| = 0.$$

Then the sequence $\|\omega_k - z_{k-1}\|^2$ is convergent. From (4.11), since the sequence $\{a_k\}$ is convergent. Thus, the sequence $\{\|\omega_k - \omega^*\|^2 + \frac{2(2\sigma-1)}{\sigma}\beta F(\omega^*)^\top(\omega_{k-1}-\omega^*)\}$ is convergent. In the following, we will prove that $\{\omega^k\}$ globally converges to $\hat{\omega}$.

Assume that this conclusion is false. Then there exist two subsequences $\{\omega_{k_i}\}$ and $\{\omega_{m_j}\}$ of $\{\omega_k\}$ with $\lim_{k_i\to\infty}\omega_{k_i} = \hat{\omega} \in \Omega^*$ and $\lim_{m_j\to\infty}\omega_{m_j} = \tilde{\omega} \in \Omega^*$, and $\tilde{\omega} \neq \tilde{\omega}$. Similar to discussion above, we have that both the sequence $\{\|\omega_k - \hat{\omega}\|^2 + \frac{2(2\sigma-1)}{\sigma}\beta F(\hat{\omega})^\top(\omega_{k-1}-\hat{\omega})\}$ and $\{\|\omega_k - \tilde{\omega}\|^2 + \frac{2(2\sigma-1)}{\sigma}\beta F(\tilde{\omega})^\top(\omega_{k-1}-\tilde{\omega})\}$ are convergent. By a direct computation yields that

$$\begin{array}{rl} & \lim_{k\to\infty}\|\omega_k - \hat{\omega}\|^2 + \frac{2(2\sigma-1)}{\sigma}\beta F(\hat{\omega})^\top(\omega_{k-1}-\hat{\omega}) \\ = & \lim_{k_i\to\infty}\|\omega_{k_i} - \hat{\omega}\|^2 + \frac{2(2\sigma-1)}{\sigma}\beta F(\hat{\omega})^\top(\omega_{k_i-1}-\hat{\omega}) \\ = & \liminf_{k_i\to\infty}\|\omega_{k_i} - \hat{\omega}\|^2 + \frac{2(2\sigma-1)}{\sigma}\beta F(\hat{\omega})^\top(\omega_{k_i-1}-\hat{\omega}) \\ < & \liminf_{k_i\to\infty}\|\omega_{k_i} - \tilde{\omega}\|^2 + \frac{2(2\sigma-1)}{\sigma}\beta F(\tilde{\omega})^\top(\omega_{k_i-1}-\tilde{\omega}) \\ = & \lim_{k_i\to\infty}\|\omega_{k_i} - \tilde{\omega}\|^2 + \frac{2(2\sigma-1)}{\sigma}\beta F(\tilde{\omega})^\top(\omega_{k_i-1}-\tilde{\omega}) \\ = & \lim_{k\to\infty}\|\omega_k - \tilde{\omega}\|^2 + \frac{2(2\sigma-1)}{\sigma}\beta F(\tilde{\omega})^\top(\omega_{k-1}-\tilde{\omega}) \end{array} \tag{4.14}$$

Similar to discussion in (4.14), we can also prove that

$$\lim_{k\to\infty}\|\omega_k - \tilde{\omega}\|^2 + \frac{2(2\sigma-1)}{\sigma}\beta F(\tilde{\omega})^\top(\omega_{k-1}-\tilde{\omega}) < \lim_{k\to\infty}\|\omega_k - \hat{\omega}\|^2 + \frac{2(2\sigma-1)}{\sigma}\beta F(\hat{\omega})^\top(\omega_{k-1}-\hat{\omega}). \tag{4.15}$$

Combining (4.14) with (4.15), This is contradiction, and the desired result follows. □

Theorem 4.5. *The sequence $\{x_k\}$ converges globally to a solution of (1.1), where $x_k = \mu_k - \nu_k$, $(\mu_k;\nu_k) = \omega_k$.*

Proof. From Theorem 4.4, we know that

$$\lim_{k\to\infty}\omega_k = \hat{\omega} := (\hat{\mu};\hat{\nu}). \tag{4.16}$$

Letting $\hat{x} = \hat{\mu} - \hat{\nu}$. A direct computation yields that

$$\begin{array}{rl} \|x_k - \hat{x}\| & = \|(\mu_k - \nu_k) - (\hat{\mu} - \hat{\nu})\| \\ & \leq \|(\mu_k - \hat{\mu})\| + \|(\nu_k - \hat{\nu})\| \\ & \leq \|(\mu_k - \hat{\mu})\|_1 + \|(\nu_k - \hat{\nu})\|_1 \\ & = \|(\mu_k - \hat{\mu};\nu_k - \hat{\nu})\|_1 \\ & \leq \sqrt{2n}\|(\mu_k - \hat{\mu};\nu_k - \hat{\nu})\| \\ & = \sqrt{2n}\|\omega_k - \hat{\omega}\| \to 0(\text{as} \quad k\to\infty), \end{array} \tag{4.17}$$

where the second and third inequalities follow from the fact that

$$\|x\| \leq \|x\|_1 \leq \sqrt{n}\|x\|, \forall x \in R^n.$$

Thus, The sequence $\{x_k\}$ converges globally to a solution of (1.1). □

5 Numerical Experiments

In this section, we present some numerical experiments about compressive sensing to prove the efficiency of proposed method. All codes are written by version of Matlab 9.20.538062 and performed on a Windows 7 PC with AMD FX-7500 Redaon R7, 10 compute Cores 4C+6G, 2.10GHz and 4GB of memory. For experiments, we set $n = 2^{11}$, $m = \text{floor}(n/a)$, $k = \text{floor}(m/b)$, and the matrix A is generated by Matlab scripts:

`[Q, R]=qr(A',0); A=Q'.`

The original signal $\bar{x}$ is generated by `p=randperm(n); x(p(1:k))=randn(k,1)`. Then, the observed signal is $y = A\bar{x} + \bar{n}$, where $\bar{n}$ is generated by a standard Gaussian distribution $N(0,1)$ and then it is normalized. The initial points $\omega_0 = (\mu_0; \nu_0)$, where $\mu_0 = \max\{0, A^\top y\}$, $\nu_0 = \max\{0, -A^\top y\}$. The stop criterion is

$$\frac{\|f_k - f_{k-1}\|}{\|f_{k-1}\|} < 10^{-5},$$

where f_k denotes the objective value of (1.1) at iteration x_k. We calculate the relative error

$$\text{RelErr} = \frac{\|\tilde{x} - \bar{x}\|}{\|\bar{x}\|},$$

where $\tilde{x}$ denotes the recovery signal.

Applying Algorithm 4.1, the original signal, the measurement and the recovery signal (marked by red point) is given in Fig. 1. From Fig. 1, all the original signals are circled by the red points, which indicate that the Algorithm 4.1 can recover the original signal quite well. In Tables 1, we report the number of iterations, the CPU time in seconds, the relative error of the Algorithm 4.1.

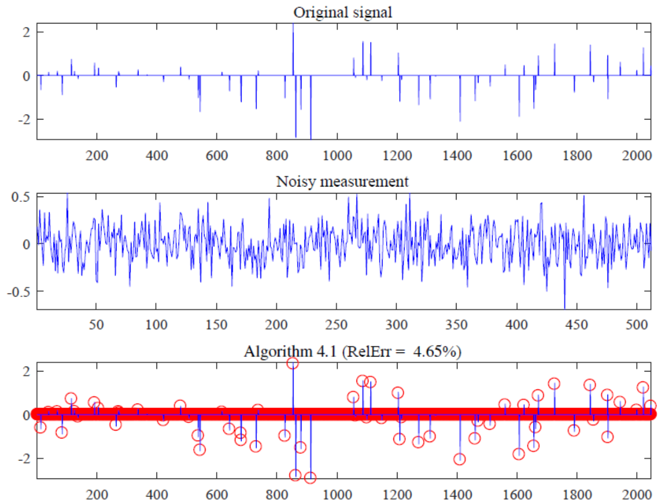

Fig. 1. The original signal, noisy measurement and recovery signal

Table 1. Results of algorithm 4.1

σ	Time	Iter	RelErr
1.0001	5.148	572	0.0462
1.01	4.586	496	0.0483
1.1	5.382	534	0.0457
10.1	4.336	478	0.0448
100.1	3.868	431	0.0441
1000.1	6.208	566	0.0501

6 Conclusion

In this paper, we propose a new projection-type algorithm for solving the compressive sensing (CS) with a closed form, and its global convergence is established in detail. Furthermore, some numerical results illustrate that the method is efficient for the given tests.

This work has several possible extensions. First, the parameters σ of Algorithm 4.1 is adjusted dynamically to further enhance the efficiency of the corresponding method. Second, how to extend Algorithm 4.1 to nonlinear variational inequalities is worthy of research.

Acknowledgement

The authors gratefully acknowledge the valuable comments of the editor and the anonymous reviewers. This work is supported by the foundation of National Natural Science Foundation of China (No. 11671228), and the Applied Mathematics Enhancement Program of Linyi University.

Competing Interests

The authors declare that no competing interests exist.

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