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Asymptotic Distribution of Unit Root Tests Base on ESTAR Model with Flexible Fourier Form

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, the asymptotic distribution of Fourier ESTAR model (FKSS) proposed by [1], which was not given in the original paper are derived. Result shows that the asymptotic distributions are functions of brownian motion, only depends on K and free from nuisance parameters.

Keywords: Structural break; Nonlinear unit root tests; Flexible Fourier form.

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1 Introduction

In the recent years, a large body of time series literatures that use Fourier approximation of unknown functional forms have emerged (see [2],[3] and [4]). Moreover, [5] propose Fourier approximation which is sufficient to approximate a wide range of functional forms, since the advantage of the Fourier approach to capture the behavior of a deterministic function of unknown form works better than dummy variable method proposed by [6] irrespective of the breaks are instantaneous or smooth.

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After that, unit root test based on nonlinear deterministic component has been widely concerned.[7] adopt the Lagrange multiplier methodology by [8] and develop a unit root test using Fourier form approximation. Similarly, [9] develop a unit root with a Fourier function in the deterministic term in a Dickey fuller type regression frame work. Furthermore, [10] develop the generalize least square unit root test proposed by [11] to allow for a Fourier approximation to the unknown deterministic component.

Considering ESTAR model with Fourier form that capture nonlinear adjustment and structural breaks well, [1] develop a new tests procedure for unit roots base on ESTAR model with flexible Fourier form.The main constraints is that [1] failed to give the asymptotic distribution of their proposed test. However, our main concern in this paper is to derive the asymptotic distribution of Fourier-ESTAR model(FKSS) proposed by [1]. The corresponding asymptotic distribution are given in next part. Result show that our derived asymptotic distributions will provides a foundation for the derivation of complex model with Fourier function, which is uncorrelated with nuisance parameter.

2 Asymptotic Properties of the Test Statistics

According to [5], a single-frequency Component of a Fourier approximation can mimic a wide variety of breaks and other types of non-linearity we begin our analysis with a Data generating process containing only one frequency.

$$y_t = \alpha_0 + \alpha_1 t + \alpha_2 \sin(\frac{2\pi kt}{T}) + \alpha_3 \cos(\frac{2\pi kt}{T}) + v_t, \tag{2.1}$$

$$v_t = \rho v_{t-1} + \gamma v_{t-1}(1 - \exp(-\theta v_{t-1}^2)) + \epsilon_t. \tag{2.2}$$

where $\epsilon_t \sim iid(0, \sigma^2)$, k represent a particular frequency and T is the sample Size

In this study, two step testing procedure are used to derive the asymptotic distribution of the Fourier-ESTAR model.

Remark 2.1: The deterministic kernel considered in equation(2.1) includes a linear time trend, but we may also consider the case where only a constant and a Fourier terms are considered; i.e., the case where $\alpha_1 = 0$ in equation(2.1).This will be referred to demeaned case in what follows, while the more general case $\alpha_1 \neq 0$ will be termed the detrended case.

In the first step, we obtaine the demeaned and detrended series of equation(1) as follows:

<u>Demeaned Case, $\alpha_1 = 0$:</u>

$$\tilde{v_t} = y_t - \hat{\alpha_0} - \hat{\alpha_2} \sin(\frac{2\pi kt}{T}) - \hat{\alpha_3} \cos(\frac{2\pi kt}{T}),$$

This can be re-written as:

$$\tilde{v_t} = y_t - Z_t'(\hat{\theta}). \tag{2.3}$$

where $\theta = (\alpha_0, \alpha_2, \alpha_3)'$, $\hat{\theta}$ is the OLS estimate of θ and $Z_t = (1, \sin(\frac{2\pi kt}{T}), \cos(\frac{2\pi kt}{T}))'$

Detrended Case, $\alpha_1 \neq 0$:

$$\tilde{v_t} = y_t - \hat{\alpha_0} - \hat{\alpha_1} t - \hat{\alpha_2}\sin(\frac{2\pi kt}{T}) - \hat{\alpha_3}\cos(\frac{2\pi kt}{T}),$$

This can be re-written as:

$$\tilde{v_t} = y_t - Z_t'(\hat{\theta}). \tag{2.4}$$

where $\theta = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)'$, $\hat{\theta}$ is the OLS estimate of θ and $Z_t = (1, t, \sin(\frac{2\pi kt}{T}), \cos(\frac{2\pi kt}{T}))'$

In the second step (section(2.1)) we adopt a Fourier base unit root test (KSS test) with demeaned or detrended and run the ESTAR-type unit root test.

2.1 KSS test

To construct a fourier base unit root test we use classical KSS unit root. The ESTAR model in equation(2.2) will be reparametrized with $\tilde{v_t}$ instead of v_t as follows:

$$\tilde{v_t} = \rho\tilde{v}_{t-1} + \gamma\tilde{v}_{t-1}(1 - \exp(-\theta\tilde{v}_{t-1}^2)) + \epsilon_t. \tag{2.5}$$

equation(2.5) can be written as:

$$\Delta\tilde{v_t} = \phi\tilde{v}_{t-1} + \gamma\tilde{v}_{t-1}(1 - \exp(-\theta\tilde{v}_{t-1}^2)) + \epsilon_t. \tag{2.6}$$

Where $\phi = \rho - 1$

Following the practice in the literature (e.g. [12] in the context of TAR models and [13] in the context of ESTAR models), we impose $\phi = 0$ in equation(2.6), implying that $\tilde{v_t}$ follows a unit root process in the middle regime. we can write equation(2.6) as :

$$\Delta\tilde{v_t} = \gamma\tilde{v}_{t-1}(1 - \exp(-\theta\tilde{v}_{t-1}^2)) + \epsilon_t. \tag{2.7}$$

equation (2.7) implies that $\tilde{v_t}$ follows either a unit root or globally stationary, we consider testing the null hypothesis that $\tilde{v_t}$ follow a unit root process given by $\gamma = 0$ or $\theta = 0$, against the alternative that $\tilde{v_t}$ is nonlinear and globally stationary, i.e $\theta = 0$ with $-2 < \gamma < 0$.

Obviously, testing the null hypothesis $H_o : \theta = 0$ in equation(2.7) directly is not feasible, since γ is not identified under the null hypothesis. See for example [14]. A popular approach to avoid the presence of nuisance parameters under the null hypothesis is to use a Taylor approximation of the smooth transition function $G(\tilde{v}_{t-1}; \theta) = 1 - \exp(-\theta\tilde{v}_{t-1}^2)$ around $\theta = 0$ see [15]. An application of a first-order Taylor approximation to the ESTAR model leads to the auxiliary equation below:

$$\Delta\tilde{v_t} = \delta\tilde{v}_{t-1}^3 + \epsilon_t. \tag{2.8}$$

The unit root hypothesis is set up by estimating equation(2.8) with OLS and testing the null $H_0 : \delta = 0$ against the alternative $H_1 : \delta < 0$ using t-statistics define as:

$$t_i^{kss} = \frac{\hat{\delta}}{s.e(\hat{\gamma})}. \tag{2.9}$$

where $\hat{\delta}$ is the OLS estimate of δ in equation(2.8), $s.e(\hat{\delta})$ is the corresponding standard error, and $i = (\mu, \tau)$ for demeaned and detrended cases respectively.

To obtained the asymptotic distribution of the Fourier ESTAR model define in equation(2.9), the following result are needed.

Proposition 2.1.

i $\frac{1}{T^{3/2}}\sum_{k=1}^{T} y_t \implies \sigma\int_0^1 W(r)dr = \sigma f_1$

ii $\frac{1}{T^{5/2}}\sum_{k=1}^{T} ty_t \implies \sigma\int_0^1 rW(r)dr = \sigma f_2$

iii $\frac{1}{T^{3/2}}\sum_{k=1}^{T} \sin(\frac{2\pi kt}{T})y_t \implies \sigma\int_0^1 \sin(2\pi kr)W(r)dr = \sigma f_3$

iv $\frac{1}{T^{3/2}}\sum_{k=1}^{T} \cos(\frac{2\pi kt}{T})y_t \implies \sigma\int_0^1 \cos(2\pi kr)W(r)dr = \sigma f_4$

v $\frac{1}{T}\sum_{k=1}^{T} \sin(\frac{2\pi kt}{T}) \implies 0$

vi $\frac{1}{T}\sum_{k=1}^{T} \cos(\frac{2\pi kt}{T}) \implies 0$

vii $\frac{1}{T}\sum_{k=1}^{T} \sin^2(\frac{2\pi kt}{T}) \implies 1/2$

viii $\frac{1}{T}\sum_{k=1}^{T} \cos^2(\frac{2\pi kt}{T}) \implies 1/2$

ix $\frac{1}{T^2}\sum_{k=1}^{T} t\sin(\frac{2\pi kt}{T}) \implies \frac{-1}{(2\pi k)}$

x $\frac{1}{T^2}\sum_{k=1}^{T} t\cos(\frac{2\pi kt}{T}) \implies 0$

xi $\frac{1}{T}\sum_{k=1}^{T} \cos(\frac{2\pi kt}{T})\sin(\frac{2\pi kt}{T}) \implies 0$

Theorem 2.1. *Under the null hypothesis the t-statistics defined in equation(2.9), for the demeaned case has the following asymptotic distribution:*

$$t_{\mu}^{F-kss} \implies \frac{\int_0^1 W_{\mu}(k,r)^3 dw(r)}{(\int_0^1 W_{\mu}(k,r)^6 dr)^{1/2}}.$$

where $W_{\mu}(k,r)$ is demeaned Brownian motion defined on $r \in (0,1)$

Theorem 2.2. *Under the null hypothesis the t-statistics defined in equation(2.9), for the detrended case has the following asymptotic distribution:*

$$t_{\tau}^{F-kss} \implies \frac{\int_0^1 W_{\tau}(k,r)^3 dw(r)}{(\int_0^1 W_{\tau}(k,r)^6 dr)^{1/2}}.$$

where $W_{\tau}(k,r)$ is detrended Brownian motion defined on $r \in (0,1)$

The asymptotic distribution of the test-statistics will only depends on K and free from nuisance parameter.

Proof. See the Appendix.

3 Conclusions

[1] Focus on the potential effect that structural breaks and non-linear mean reversion have on tests of the Purchasing Power Parity (PPP) hypothesis. They present tests that, far from considering these two features separately, model both breaks and non-linear adjustment jointly,but the main constrains is that [1] do not give asymptotic distributions of there test. This article extend the work of [1] by providing the asymptotic distributions of Fourier-ESTAR model which is not available in the original paper.

Competing Interests

Authors have declared that no competing interests exist.

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Appendix

Proof of Proposition:

From Proposition (2.1) under null hypothesis $\Delta y_t = \epsilon_t$

i $\frac{1}{T^{3/2}} \sum_{k=1}^{T} y_t \implies \sigma \int_0^1 W(r)dr = \sigma f_1$

ii $\frac{1}{T^{5/2}} \sum_{k=1}^{T} t y_t \implies \sigma \int_0^1 rW(r)dr = \sigma f_2$

The result of (i) and (ii) are standard, (see [16]). from the continuous mapping theorem , we have:

iii $\frac{1}{T^{3/2}} \sum_{k=1}^{T} \sin(\frac{2\pi kt}{T}) y_t \implies \sigma \int_0^1 \sin(2\pi kr)W(r)dr = \sigma f_3$

iv $\frac{1}{T^{3/2}} \sum_{k=1}^{T} \cos(\frac{2\pi kt}{T}) y_t \implies \sigma \int_0^1 \cos(2\pi kr)W(r)dr = \sigma f_4$

v $\frac{1}{T} \sum_{k=1}^{T} \sin(\frac{2\pi kt}{T}) \implies \int_0^1 \sin(2\pi kr)dr = \frac{1}{2\pi k}(1 - \cos(2\pi kr)) = 0$

vi $\frac{1}{T} \sum_{k=1}^{T} \cos(\frac{2\pi kt}{T}) \implies \int_0^1 \cos(2\pi kr)dr = \frac{\sin(2\pi k)}{2\pi k} = 0$

vii $\frac{1}{T} \sum_{k=1}^{T} \sin^2(\frac{2\pi kt}{T}) \implies \int_0^1 \sin^2(2\pi kr)dr = \frac{1}{2}\int_0^1 (1 - \cos(4\pi kr))dr = \frac{1}{2} - \frac{\sin 4\pi k}{4\pi k} = 1/2$

viii $\frac{1}{T} \sum_{k=1}^{T} \cos^2(\frac{2\pi kt}{T}) \implies \int_0^1 \cos^2(2\pi kr)dr = \frac{1}{2}\int_0^1 (1 - \sin^2(4\pi kr))dr = \frac{1}{2} + \frac{\sin 4\pi k}{4\pi k} = 1/2$

ix $\frac{1}{T^2} \sum_{k=1}^{T} t\sin(\frac{2\pi kt}{T}) \implies \int_0^1 r\sin(2\pi kr)dr = \frac{\sin(2\pi k)}{(2\pi k)^2} - \frac{\cos(2\pi k)}{(2\pi k)} = \frac{-1}{(2\pi k)}$

x $\frac{1}{T^2} \sum_{k=1}^{T} t\cos(\frac{2\pi kt}{T}) \implies \int_0^1 r\cos(2\pi kr)dr = \frac{\cos(2\pi k) - 1}{(2\pi k)^2} + \frac{\sin(2\pi k)}{(2\pi k)} = 0$

xi $\frac{1}{T} \sum_{k=1}^{T} \cos(\frac{2\pi kt}{T})\sin(\frac{2\pi kt}{T}) \implies \int_0^1 \cos(2\pi kr)\sin(2\pi kr)dr = \frac{1 - \cos(4\pi k)}{8\pi k} = 0$

Proof of Theorem 2.1:

Proof. Consider the level of stationarity with Fourier Function. for the demeaned case i.e $\alpha_1 = 0$ in equation(2.1), by using the OLS residual define in equation(2.6) with $Z_t = (1, \sin(\frac{2\pi kt}{T}), \cos(\frac{2\pi kt}{T}))$. define as follows: □

$$\tilde{v_t} = y_t - Z_t'(\hat{\theta})$$

where $\theta = (\alpha_0, \alpha_2, \alpha_3)'$ and $\hat{\theta}$ is the OLS estimate of θ, and $\Delta y_t = \epsilon_t$. Let $z = (z_1, ...z_t)'$ and $y = (y_1, ...y_t)'$ also we define the scalling parameter as $\Upsilon = diag[\frac{1}{\sqrt{T}}, \frac{1}{\sqrt{T}}, \frac{1}{\sqrt{T}}]$ then we have:

$$\Upsilon(\hat{\theta}) = [\Upsilon^{-1}(Z'Z)\Upsilon^{-1}]^{-1}\Upsilon^{-1}Z'Y$$

from the above equation we show the asymptotic distributon of the demeaned case as follows:

$$\frac{1}{\sqrt{T}}\tilde{v}_{t[Tr]} = \frac{1}{\sqrt{T}}y_{[Tr]} - \frac{1}{\sqrt{T}}Z'_{[Tr]}(\hat{\theta})$$

$$\frac{1}{\sqrt{T}}\tilde{v}_{t[Tr]} = \frac{1}{\sqrt{T}}y_{[Tr]} - \frac{1}{\sqrt{T}}Z'_{[Tr]}\Upsilon^{-1}[\Upsilon^{-1}(Z'Z)\Upsilon^{-1}]^{-1}\Upsilon^{-1}Z'Y$$

$$\frac{1}{\sqrt{T}}\tilde{v}_{t[Tr]} = \frac{1}{\sqrt{T}}y_{[Tr]} - \frac{1}{T}Z'_{[Tr]}[\Upsilon^{-1}(Z'Z)\Upsilon^{-1}]^{-1}\Upsilon^{-1}Z'Y$$

According to central limit theorem each terms is defined as follows:

$$\frac{1}{\sqrt{T}}y_{[Tr]} = \frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} u_{[Tr]} \longrightarrow \sigma W(r)$$

$$[\Upsilon^{-1}(Z'Z)\Upsilon^{-1}]^{-1} = \begin{bmatrix} \frac{T}{T} & \frac{1}{T}\sum_{k=1}^{T}\sin(\frac{2\pi kt}{T}) & \frac{1}{T}\sum_{k=1}^{T}\cos(\frac{2\pi kt}{T}) \\ & \frac{1}{T}\sum_{k=1}^{T}\sin^2(\frac{2\pi kt}{T}) & \frac{1}{T}\sum_{k=1}^{T}\sin(\frac{2\pi kt}{T})\cos(\frac{2\pi kt}{T}) \\ & & \frac{1}{T}\sum_{k=1}^{T}\cos^2(\frac{2\pi kt}{T}) \end{bmatrix}^{-1}$$

by using the proposition define earlier we have:

$$[\Upsilon^{-1}(Z'Z)\Upsilon^{-1}]^{-1} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ & \frac{1}{2} & 0 \\ & & \frac{1}{2} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ & 2 & 0 \\ & & 2 \end{bmatrix}$$

Also

$$T^{-1}\Upsilon^{-1}Z'Y = \begin{bmatrix} \frac{1}{T^{1/2}}\sum_{k=1}^{T} y_t \\ \frac{1}{T^{1/2}}\sum_{k=1}^{T}\sin(\frac{2\pi kt}{T})y_t \\ \frac{1}{T^{1/2}}\sum_{k=1}^{T}\cos(\frac{2\pi kt}{T})y_t \end{bmatrix} \longrightarrow \begin{bmatrix} \sigma f_1 \\ \sigma f_3 \\ \sigma f_4 \end{bmatrix}$$

Then we can write that:

$$\frac{1}{T}Z'_{[Tr]}\Upsilon^{-1}[\Upsilon^{-1}(Z'Z)\Upsilon^{-1}]^{-1}\Upsilon^{-1}Z'Y \longrightarrow \begin{bmatrix}1 & \sin(2\pi kr) & \cos(2\pi Kr)\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}\sigma f_1\\ \sigma f_3\\ \sigma f_4\end{bmatrix}$$

$$\longrightarrow \begin{bmatrix}\sigma f_1 + 2\sin(2\pi kr)\sigma f_3 + 2\cos(2\pi kr)\sigma f_4\end{bmatrix}$$

by combining all the result we obtain the demeaned brownian motion as:

$$\frac{1}{\sqrt{T}}\tilde{v_t}_{[Tr]} = \frac{1}{\sqrt{T}}y_{[Tr]} - \frac{1}{\sqrt{T}}Z'_{[Tr]}(\hat{\theta})$$

$$\frac{1}{\sqrt{T}}\tilde{v_t}_{[Tr]} \longrightarrow \sigma[W(r) - f_1 - 2\sin(2\pi kr)f_3 - 2\cos(2\pi kr)f_4] \equiv \sigma W_\mu(kr)$$

$$\frac{1}{\sigma\sqrt{T}}\tilde{v_t}_{[Tr]} \longrightarrow W_\mu(kr)$$

using above results, under the null we can obtain that:

$$t_\mu^{F-kss} \longrightarrow \frac{\int_0^1 W_\mu(k,r)^3 dw(r)}{(\int_0^1 W_\mu(k,r)^6 dr)^{1/2}}.$$

Proof of Theorem 2.2:

Proof. Consider the level of sattionarity with Fourier Function. for the detrended case i.e $\alpha_1 \neq 0$ in equation(2.1), by using the OLS residual define in equation(2.7) with $Z_t = (1, t, \sin(\frac{2\pi kt}{T}), \cos(\frac{2\pi kt}{T}))$. define as follows:

$$\tilde{v_t} = y_t - Z'_t(\hat{\theta})$$

where $\theta = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)'$ and $\hat{\theta}$ is the OLS estimate of θ and$\Delta y_t = \epsilon_t$. Let $z = (z_1, ...z_t)'$ and $y = (y_1, ...y_t)'$ also we define the scalling parameter as $\Upsilon = diag[\frac{1}{\sqrt{T}}, \frac{1}{T^{1.5}}, \frac{1}{\sqrt{T}}, \frac{1}{\sqrt{T}}]$ then we have:

$$\Upsilon(\hat{\theta}) = [\Upsilon^{-1}(Z'Z)\Upsilon^{-1}]^{-1}\Upsilon^{-1}Z'Y$$

from the above equation we show the asymptotic distributon of the detrended case as follows:

$$\frac{1}{\sqrt{T}}\tilde{v_t}_{[Tr]} = \frac{1}{\sqrt{T}}y_{[Tr]} - \frac{1}{T}Z'_{[Tr]}[\Upsilon^{-1}(Z'Z)\Upsilon^{-1}]^{-1}\Upsilon^{-1}Z'Y$$

following the same procedure discused earlier and according to functional central limit theorem we have terms as follows:

$$\frac{1}{\sqrt{T}}y_{[Tr]} = \frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]}u_{[Tr]} \longrightarrow \sigma W(r)$$

$$[\Upsilon^{-1}(Z'Z)\Upsilon^{-1}]^{-1} = \begin{bmatrix} \frac{T}{T} & \frac{1}{T^2}\sum_{k=1}^{T} t & \frac{1}{T}\sum_{k=1}^{T}\sin(\frac{2\pi kt}{T}) & \frac{1}{T}\sum_{k=1}^{T}\cos(\frac{2\pi kt}{T}) \\ & \frac{1}{T^3}\sum_{k=1}^{T} t^2 & \frac{1}{T}\sum_{k=1}^{T} t\sin(\frac{2\pi kt}{T}) & \frac{1}{T}\sum_{k=1}^{T} t\sin(\frac{2\pi kt}{T}) \\ & & \frac{1}{T}\sum_{k=1}^{T}\sin^2(\frac{2\pi kt}{T}) & \frac{1}{T}\sum_{k=1}^{T}\sin(\frac{2\pi kt}{T})\cos(\frac{2\pi kt}{T}) \\ & & & \frac{1}{T}\sum_{k=1}^{T}\cos^2(\frac{2\pi kt}{T}) \end{bmatrix}^{-1}$$

by using the proposition define earlier we have:

$$[\Upsilon^{-1}(Z'Z)\Upsilon^{-1}]^{-1} \longrightarrow \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{-1}{2\pi K} & 0 \\ 0 & \frac{-1}{2\pi K} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}^{-1} = \frac{1}{\frac{k^2\pi^2-6}{48k^2\pi^2}} \begin{bmatrix} \frac{2k^2\pi^2-3}{24k^2\pi^2} & \frac{-1}{8} & \frac{-1}{8k\pi} & 0 \\ \frac{-1}{8} & \frac{1}{4} & \frac{1}{4k\pi} & 0 \\ \frac{-1}{8k\pi} & \frac{1}{4k\pi} & \frac{1}{24} & 0 \\ \frac{1}{8} & \frac{-1}{4} & \frac{-1}{4k\pi} & \frac{k^2\pi^2-6}{24k^2\pi^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4k^2\pi^2-6}{k^2\pi^2-6} & \frac{-6k^2\pi^2}{k^2\pi^2-6} & \frac{-6k\pi}{k^2\pi^2-6} & 0 \\ \frac{-6k^2\pi^2}{k^2\pi^2-6} & \frac{12k^2\pi^2}{k^2\pi^2-6} & \frac{12k\pi}{k^2\pi^2-6} & 0 \\ \frac{-6k\pi}{k^2\pi^2-6} & \frac{12k\pi}{k^2\pi^2-6} & \frac{2k^2\pi^2}{k^2\pi^2-6} & 0 \\ \frac{6k^2\pi^2}{k^2\pi^2-6} & \frac{-12k^2\pi^2}{k^2\pi^2-6} & \frac{-12k\pi}{k^2\pi^2-6} & 2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

Also

$$T^{-1}\Upsilon^{-1}Z'Y = \begin{bmatrix} \frac{1}{T^{1/2}}\sum_{k=1}^{T} y_t \\ \frac{1}{T^{3/2}}\sum_{k=1}^{T} ty_t \\ \frac{1}{T^{1/2}}\sum_{k=1}^{T}\sin(\frac{2\pi kt}{T})y_t \\ \frac{1}{T^{1/2}}\sum_{k=1}^{T}\cos(\frac{2\pi kt}{T})y_t \end{bmatrix} \longrightarrow \begin{bmatrix} \sigma f_1 \\ \sigma f_2 \\ \sigma f_3 \\ \sigma f_4 \end{bmatrix}$$

Then we can write that:

$$\frac{1}{T}Z'_{[Tr]}\Upsilon^{-1}[\Upsilon^{-1}(Z'Z)\Upsilon^{-1}]^{-1}\Upsilon^{-1}Z'Y \longrightarrow \begin{bmatrix}1 & r & \sin(2\pi kr) & \cos(2\pi Kr)\end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \begin{bmatrix} \sigma f_1 \\ \sigma f_2 \\ \sigma f_2 \\ \sigma f_3 \end{bmatrix}$$

$$\longrightarrow \sigma \begin{Bmatrix} (a_{11}f_1 + a_{12}f_2 + a_{13}f_3) + (a_{21}f_1 + a_{22}f_2 + a_{23}f_3)r \\ +(a_{31}f_1 + a_{32}f_2 + a_{22}f_3)sin(2\pi kr) + a_{44}f_4cos(2\pi kr) \end{Bmatrix})$$

by combining all the result we obtain the detrended brownian motion as:

$$\frac{1}{\sqrt{T}}\tilde{v}_{t[Tr]} = \frac{1}{\sqrt{T}}y_{[Tr]} - \frac{1}{\sqrt{T}}Z'_{[Tr]}(\hat{\theta})$$

$$\frac{1}{\sqrt{T}}\tilde{v}_{t[Tr]} \longrightarrow \sigma(W(r) - \sigma \begin{bmatrix} (a_{11}f_1 + a_{12}f_2 + a_{13}f_3) + (a_{21}f_1 + a_{22}f_2 + a_{23}f_3)r \\ +(a_{31}f_1 + a_{32}f_2 + a_{22}f_3)sin(2\pi kr) + a_{44}f_4cos(2\pi kr) \end{bmatrix} \equiv \sigma W_\tau(kr)$$

$$\frac{1}{\sigma\sqrt{T}}\tilde{v}_{t[Tr]} \longrightarrow W_\tau(kr)$$

using above results, under the null we can obtain that:

$$t_\tau^{F-kss} \longrightarrow \frac{\int_0^1 W_\mu(k,r)^3 dw(r)}{(\int_0^1 W_\mu(k,r)^6 dr)^{1/2}}.$$

□

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