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SACF of the Errors of Stationary Time Series Models in the Presence of a Large Additive Outlier

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this paper, the limiting behaviour of the Sample Autocorrelation Function(SACF) of the errors $\{e_t\}$ of First-Order Autoregressive (AR(1)), First-Order Moving Average (MA(1)) and First Order Autoregressive First-Order Moving Average (ARMA(1,1)) stationary time series models in the presence of a large Additive Outlier(AO) is discussed. It is found that the errors which are supposed to be uncorrelated due to either white noise process or normally distributed process are not so in the presence of a large additive outlier. The SACF of the errors follows a particular pattern based on the time series model. In the case of AR(1) model, at lag 1, the contaminated errors $\{e_t\}$ are correlated, whereas at higher lags, they are uncorrelated. But in the MA(1) and ARMA(1,1) models, the contaminated errors $\{e_t\}$ are correlated at all the lags. Furthermore it is observed that the intensity of correlations depends on the parameters of the respective models.

Keywords: SACF; Errors; AR(1); MA(1); ARMA(1,1); Additive Outlier.

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1 Introduction

Outliers can result for many external or internal reasons. Measurement (recording or typing) errors, classification mistakes in sampling or some non-repetitive exogenous interventions can have effects in the form of outliers (isolated or patchy). Economic and business time series are sometimes subject to the influence of strikes, outbreaks of wars, sudden change in the market structure of some group of commodities, technical change or new equipment in a communication system, or simply unexpected pronounced changes in weather etc. These unusual observations can significantly affect the methodology of time series analysis. The presence of outliers can lead to model misspecification, biased parameter estimation, poor forecasts, etc. Hafner and Preminger [1] have investigated the impact of outliers (especially additive outlier) on the fractional unit root test. Ledolter [2] discusses the effect of additive outliers on the forecasts from time series models. The effect of the outliers on the sampling distribution of the estimated parameters of a time series model is explored by Urooj and Asghar [3]. It is important to investigate further to understand their impact on other aspects of time series analysis.

In time series analysis, we either assume that the errors $\{a_t\}$ are white noise process or a sequence of independently and identically distributed normal random variables. In either case, it implies that the errors are uncorrelated. This assumption would be satisfied, if there are no outliers. It would be interesting to explore the same when there exists outliers. In this article, we focus on the limiting behaviour of Sample Autocorrelation Function (SACF) of the errors of stationary time series models: First-Order Autoregressive (AR(1)), First-Order Moving Average Process (MA(1)) and First Order Autoregressive First-Order Moving Average (ARMA(1,1)) in the presence of a large Additive Outlier (AO). The motivation for this is due to Chan [4], who discusses the effect of Additive Outlier (AO), Innovational Outlier (IO), Level Shift (LS) and Temporary Change (TC) on the SACF of the series. Further, Maronna et al. [5] have discussed the effect of a Patch of AO's on the first order SACF of the series. The same has been extended by Suresh [6] to higher orders. Further, the limiting behaviour of the SACF and Sample Partial Autocorrelation Function (SPACF) in the presence of a Doublet Outlier is also discussed by Suresh [6]. In this article, along the lines of Chan [4] and Maronna et al. [5], the impact of a large AO on the SACF of the errors will be explored.

Outliers can take several forms in time series. The formal definitions and a classification of outliers in a time series context were first proposed by Fox [7]. One can resort to Tsay [8] to know more about other types of outliers in time series. When outliers or structural changes occur, $\{X_t\}$ which is a time series gets disturbed and is unobservable. We assume that the series $\{X_t\}$ follows Box-Jenkins univariate time series model [9]. In this case, it is assumed that the observed series $\{Y_t\}$ follows the model

$$Y_t = f(t) + X_t \quad (1.1)$$

where $f(t)$ is a parametric function representing the exogenous disturbances of X_t such as outliers or level changes. The function $f(t)$ may be deterministic or stochastic depending on the types of disturbances. In practice, $f(t)$ is specified by data analysts based on the substantive information of the disturbances and the process $\{Y_t\}$. For the deterministic model, it is assumed that $f(t)$ is of the form

$$f(t) = \delta_0 \frac{\omega(B)}{\delta(B)} I_t^T \quad (1.2)$$

where

$$I_t^T = \begin{cases} 1 & \text{if } t = T, \\ 0 & \text{if } t \neq T, \end{cases} \quad (1.3)$$

is an indicator variable signifying the occurrence of a disturbance at the time point T , $\omega(B) = 1 - \omega_1 B - \omega_2 B^2 - \dots - \omega_s B^s$ and $\delta(B) = 1 - \delta_1 B - \delta_2 B^2 - \dots - \delta_r B^r$ are polynomials in B of

degrees s and r , respectively, and δ_0 is a constant denoting the initial impact of the disturbance. A special case of $f(t)$ is Additive Outlier.

Definition 1.1. Additive Outlier(AO): If $\delta_0 = \delta$ and $\frac{\omega(B)}{\delta(B)} = 1$ in (1.2), the model turns out to be an Additive Outlier (AO) model. It is the type of outliers that affects a single observation X_T . After this disturbance, the series returns to its normal path as if nothing has happened. A typical example of AO is the recording error. The effect caused by AO at time $t = T$, with the magnitude of the effect denoted by δ is given by

$$Y_t = X_t + \delta I_t^T. \quad (1.4)$$

The paper is structured as follows: In section 2, the impact of a large AO on the SACF of the errors of the time series models: AR(1), MA(1) and ARMA(1,1) is investigated. Finally, the discussion and conclusions is presented in section 3.

2 Sample Autocorrelation Function (SACF) of the Errors of Time Series in the Presence of an Additive Outlier

Let $\{X_t; 1, 2, \dots, n\}$ be any time series data and $\{Y_t; 1, 2, \dots, n\}$ be contaminated data obtained by superimposing a AO of magnitude of δ at $t = T$ on $\{X_t\}$. Suppose, we denote the contaminated error series as $\{e_t; 1, 2, \dots, n\}$, the following theorems in subsections 2.1 - 2.3 provides the limiting behaviour of the SACF of $\{e_t; 1, 2, \dots, n\}$ as δ tends to infinity under the AR(1), MA(1) and ARMA(1,1) models respectively.

2.1 SACF of the Errors of AR(1) model

Suppose the time series $\{X_t; 1, 2, \dots, n\}$ follow an AR(1) model, which is defined as below.

Definition 2.1 (AR(1) Process). : $\{X_t\}$ is an AR(1) process if $\{X_t\}$ is stationary and if for every t , which is defined as

$$X_t - \phi X_{t-1} = a_t, \quad (2.1)$$

where $\{a_t\} \sim WN(0, \sigma_a^2)$ or $\{a_t\} \sim N(0, \sigma_a^2)$ and ϕ is the autoregressive parameter.

Let $\{Y_t; 1, 2, \dots, n\}$ be contaminated data obtained by superimposing an AO of magnitude of δ at $t = T$ on $\{X_t\}$.

Theorem 2.1. Let us denote the SACF by $\hat{\rho}_k$, $k \geq 1$. Then for a fixed n

$$\lim_{\delta \rightarrow \infty} \hat{\rho}_k = \begin{cases} -\frac{\phi}{1 + \phi^2} & , k = 1, \\ 0 & , k \geq 2. \end{cases}$$

Proof. By definition, when $k = 1$,

$$\hat{\rho}_1 = \frac{\sum_{t=1}^{n-1} e_t e_{t+1}}{\sum_{t=1}^n e_t^2} \quad (2.2)$$

It is known from Janhavi & Suresh [10] the errors of the contaminated series $\{e_t\}$ in the AR(1) model is

$$e_t = \begin{cases} a_t & , t < T \\ \delta + a_t & , t = T \\ -\delta\phi + a_t & , t = T + 1 \\ a_t & , t > T. \end{cases} \quad (2.3)$$

Substituting $\{e_t; 1, 2, \dots, n\}$ in terms of $\{a_t; 1, 2, \dots, n\}$ in (2.2) by using (2.3), we get

$$\hat{\rho}_1 = \frac{\sum_{t=1}^{n-1} a_t a_{t+1} + \delta(a_{T-1} + a_{T+1} - \phi a_T - \phi a_{T+2}) - \delta^2 \phi}{\sum_{t=1}^n a_t^2 + \delta^2(1 + \phi^2) + 2\delta(a_T - \phi a_{T+1})} \quad (2.4)$$

Now, dividing both numerator and denominator of equation (2.4) by δ^2 , we get

$$\hat{\rho}_1 = \frac{\frac{\sum_{t=1}^{n-1} a_t a_{t+1}}{\delta^2} + \frac{\delta(a_{T-1} + a_{T+1} - \phi a_T - \phi a_{T+2})}{\delta^2} - \frac{\delta^2 \phi}{\delta^2}}{\frac{\sum_{t=1}^n a_t^2}{\delta^2} + \frac{\delta^2(1 + \phi^2)}{\delta^2} + \frac{2\delta(a_T - \phi a_{T+1})}{\delta^2}}$$

Taking $\lim_{\delta \rightarrow \infty}$ of the above equation, for a fixed n , we get

$$\begin{aligned} \lim_{\delta \rightarrow \infty} \hat{\rho}_1 &= \lim_{\delta \rightarrow \infty} \frac{\frac{\sum_{t=1}^{n-1} a_t a_{t+1}}{\delta^2} + \frac{(a_{T-1} + a_{T+1} - \phi a_T - \phi a_{T+2})}{\delta} - \phi}{\frac{\sum_{t=1}^n a_t^2}{\delta^2} + (1 + \phi^2) + \frac{2(a_T - \phi a_{T+1})}{\delta}} \\ &= \frac{\lim_{\delta \rightarrow \infty} \frac{\sum_{t=1}^{n-1} a_t a_{t+1}}{\delta^2} + \lim_{\delta \rightarrow \infty} \frac{(a_{T-1} + a_{T+1} - \phi a_T - \phi a_{T+2})}{\delta} - \phi}{\lim_{\delta \rightarrow \infty} \frac{\sum_{t=1}^n a_t^2}{\delta^2} + (1 + \phi^2) + \lim_{\delta \rightarrow \infty} \frac{2(a_T - \phi a_{T+1})}{\delta}} \\ \lim_{\delta \rightarrow \infty} \hat{\rho}_1 &= -\frac{\phi}{1 + \phi^2} \end{aligned}$$

Hence,

$$\lim_{\delta \rightarrow \infty} \hat{\rho}_1 = -\frac{\phi}{1 + \phi^2} \quad (2.5)$$

Similarly by definition, when $k \geq 2$,

$$\hat{\rho}_k = \frac{\sum_{t=1}^{n-k} e_t e_{t+k}}{\sum_{t=1}^n e_t^2} \quad (2.6)$$

Substituting $\{e_t; 1, 2, \dots, n\}$ in terms of $\{a_t; 1, 2, \dots, n\}$, it is easy to check that for $k \geq 2$,

$$\hat{\rho}_k = \frac{\sum_{t=1}^{n-k} a_t a_{t+k} + \delta(a_{T-k} + a_{T+k} - \phi a_{T-(k-1)} - \phi a_{T+(k+1)})}{\sum_{t=1}^n a_t^2 + \delta^2(1 + \phi^2) + 2\delta(a_T - \phi a_{T+1})}. \quad (2.7)$$

As before, dividing both numerator and denominator of equation (2.7) by δ^2 , and taking $\lim_{\delta \rightarrow \infty}$, we get

$$\begin{aligned} \lim_{\delta \rightarrow \infty} \hat{\rho}_k &= \lim_{\delta \rightarrow \infty} \frac{\frac{\sum_{t=1}^{n-k} a_t a_{t+k}}{\delta^2} + \frac{(a_{T-k} + a_{T+k} - \phi a_{T-(k-1)} - \phi a_{T+(k+1)})}{\delta}}{\frac{\sum_{t=1}^n a_t^2}{\delta^2} + (1 + \phi^2) + \frac{2(a_T - \phi a_{T+1})}{\delta}}, \quad k \geq 2 \\ &= \frac{\lim_{\delta \rightarrow \infty} \frac{\sum_{t=1}^{n-k} a_t a_{t+k}}{\delta^2} + \lim_{\delta \rightarrow \infty} \frac{(a_{T-k} - \phi a_{T-(k-1)} + \phi a_{T+k} - X_{T+(k+1)})}{\delta}}{\lim_{\delta \rightarrow \infty} \frac{\sum_{t=1}^n a_t^2}{\delta^2} + (1 + \phi^2) + \lim_{\delta \rightarrow \infty} \frac{2(a_T - \phi a_{T+1})}{\delta}} \\ \lim_{\delta \rightarrow \infty} \hat{\rho}_k &= \frac{0 + 0}{0 + (1 + \phi^2) + 0}, \quad k \geq 2 \end{aligned}$$

Therefore,

$$\lim_{\delta \rightarrow \infty} \hat{\rho}_k = 0, \quad k \geq 2 \quad (2.8)$$

The equations (2.5) and (2.8) prove the theorem. \square

2.2 SACF of the Errors of MA(1) model

Assume that the time series $\{X_t; 1, 2, \dots, n\}$ follow a MA(1) model, defined as

Definition 2.2 (MA(1) Process). : $\{X_t\}$ is a MA(1) process if for every t ,

$$X_t = a_t - \theta a_{t-1}, \quad (2.9)$$

where $\{a_t\} \sim WN(0, \sigma_a^2)$ or $\{a_t\} \sim N(0, \sigma_a^2)$ and θ is the moving average parameter.

Let $\{Y_t; 1, 2, \dots, n\}$ be contaminated data obtained by superimposing a AO of magnitude of δ at $t = T$ on $\{X_t\}$.

Theorem 2.2. Let us denote the SACF by $\hat{\rho}_k$, $k \geq 1$. Then for a fixed n

$$\lim_{\delta \rightarrow \infty} \hat{\rho}_k = \theta^k, \quad k \geq 1.$$

Proof. By definition, when $k = 1$,

$$\hat{\rho}_1 = \frac{\sum_{t=1}^{n-1} e_t e_{t+1}}{\sum_{t=1}^n e_t^2} \quad (2.10)$$

The errors of the contaminated series of the MA(1) model given by Janhavi & Suresh [10] are

$$e_t = \begin{cases} a_t & , t < T \\ \delta + a_t & , t = T \\ \delta \theta^j + a_t & , t = T + j, j = 1, 2, \dots, n - T \end{cases} \quad (2.11)$$

Substituting $\{e_t; 1, 2, \dots, n\}$ in terms of $\{a_t; 1, 2, \dots, n\}$ in (2.10) by using (2.11), we get

$$\hat{\rho}_1 = \frac{\sum_{t=1}^{n-1} a_t a_{t+1} + \delta \left(\sum_{j=0}^{n-T} \theta^j a_{T+j-1} + \sum_{j=0}^{n-(T-1)} \theta^j a_{T+j+1} \right) + \delta^2 \sum_{j=0}^{n-T} \theta^{2j+1}}{\sum_{t=1}^n a_t^2 + \delta^2 \sum_{j=0}^{n-T} \theta^{2j} + 2\delta \sum_{j=0}^{n-T} \theta^j a_{T+j}} \quad (2.12)$$

Now, dividing both numerator and denominator of equation (2.12) by δ^2 , we get

$$\hat{\rho}_1 = \frac{\frac{\sum_{t=1}^{n-1} a_t a_{t+1}}{\delta^2} + \frac{\delta \left(\sum_{j=0}^{n-T} \theta^j a_{T+j-1} + \sum_{j=0}^{n-(T-1)} \theta^j a_{T+j+1} \right)}{\delta^2} + \frac{\delta^2 \sum_{j=0}^{n-T} \theta^{2j+1}}{\delta^2}}{\frac{\sum_{t=1}^n a_t^2}{\delta^2} + \frac{\delta^2 \sum_{j=0}^{n-T} \theta^{2j}}{\delta^2} + \frac{2\delta \sum_{j=0}^{n-T} \theta^j a_{T+j}}{\delta^2}}$$

Taking $\lim_{\delta \rightarrow \infty}$ of the above equation, for a fixed n , we get

$$\begin{aligned} \lim_{\delta \rightarrow \infty} \hat{\rho}_1 &= \lim_{\delta \rightarrow \infty} \frac{\frac{\sum_{t=1}^{n-1} a_t a_{t+1}}{\delta^2} + \frac{\delta \left(\sum_{j=0}^{n-T} \theta^j a_{T+j-1} + \sum_{j=0}^{n-(T-1)} \theta^j a_{T+j+1} \right)}{\delta^2} + \sum_{j=0}^{n-T} \theta^{2j+1}}{\frac{\sum_{t=1}^n a_t^2}{\delta^2} + \frac{\delta^2 \sum_{j=0}^{n-T} \theta^{2j}}{\delta^2} + \frac{2\delta \sum_{j=0}^{n-T} \theta^j a_{T+j}}{\delta^2}} \\ &= \frac{\lim_{\delta \rightarrow \infty} \frac{\sum_{t=1}^{n-1} a_t a_{t+1}}{\delta^2} + \lim_{\delta \rightarrow \infty} \frac{\left(\sum_{j=0}^{n-T} \theta^j a_{T+j-1} + \sum_{j=0}^{n-(T-1)} \theta^j a_{T+j+1} \right)}{\delta} + \sum_{j=0}^{n-T} \theta^{2j+1}}{\lim_{\delta \rightarrow \infty} \frac{\sum_{t=1}^n a_t^2}{\delta^2} + \sum_{j=0}^{n-T} \theta^{2j} + \lim_{\delta \rightarrow \infty} \frac{2 \sum_{j=0}^{n-T} \theta^j a_{T+j}}{\delta}} \\ &= \frac{\sum_{j=0}^{n-T} \theta^{2j+1}}{\sum_{j=0}^{n-T} \theta^{2j}} \\ &= \frac{\theta + \theta^3 + \dots + \theta^{2(n-T)+1}}{1 + \theta^2 + \dots + \theta^{2(n-T)}} \\ &= \frac{\theta(1 + \theta^2 + \dots + \theta^{2(n-T)})}{1 + \theta^2 + \dots + \theta^{2(n-T)}} \\ \lim_{\delta \rightarrow \infty} \hat{\rho}_1 &= \theta \end{aligned}$$

Hence,

$$\lim_{\delta \rightarrow \infty} \hat{\rho}_1 = \theta. \quad (2.13)$$

Similarly by definition, when $k \geq 2$,

$$\hat{\rho}_k = \frac{\sum_{t=1}^{n-k} e_t e_{t+k}}{\sum_{t=1}^n e_t^2} \quad (2.14)$$

Substituting $\{e_t; 1, 2, \dots, n\}$ in terms of $\{a_t; 1, 2, \dots, n\}$, it is easy to check that for $k \geq 2$,

$$\hat{\rho}_k = \frac{\sum_{t=1}^{n-k} a_t a_{t+k} + \delta \left(\sum_{j=0}^{n-T} \theta^j a_{T+j-k} + \sum_{j=0}^{n-(T-k)} \theta^j a_{T+j+k} \right) + \delta^2 \sum_{j=0}^{n-T} \theta^{2j+k}}{\sum_{t=1}^n a_t^2 + \delta^2 \sum_{j=0}^{n-T} \theta^{2j} + 2\delta \sum_{j=0}^{n-T} \theta^j a_{T+j}}. \quad (2.15)$$

As before, dividing both numerator and denominator of equation (2.15) by δ^2 , and taking $\lim_{\delta \rightarrow \infty}$, we get

$$\begin{aligned} \lim_{\delta \rightarrow \infty} \hat{\rho}_k &= \lim_{\delta \rightarrow \infty} \frac{\frac{\sum_{t=1}^{n-k} a_t a_{t+k}}{\delta^2} + \frac{\delta \left(\sum_{j=0}^{n-T} \theta^j a_{T+j-k} + \sum_{j=0}^{n-(T-k)} \theta^j a_{T+j+k} \right)}{\delta^2} + \frac{\delta^2 \sum_{j=0}^{n-T} \theta^{2j+k}}{\delta^2}}{\frac{\sum_{t=1}^n a_t^2}{\delta^2} + \frac{\delta^2 \sum_{j=0}^{n-T} \theta^{2j}}{\delta^2} + \frac{2\delta \sum_{j=0}^{n-T} \theta^j a_{T+j}}{\delta^2}} \\ &= \frac{\lim_{\delta \rightarrow \infty} \frac{\sum_{t=1}^{n-k} a_t a_{t+k}}{\delta^2} + \lim_{\delta \rightarrow \infty} \frac{\left(\sum_{j=0}^{n-T} \theta^j a_{T+j-k} + \sum_{j=0}^{n-(T-k)} \theta^j a_{T+j+k} \right)}{\delta} + \sum_{j=0}^{n-T} \theta^{2j+k}}{\lim_{\delta \rightarrow \infty} \frac{\sum_{t=1}^n a_t^2}{\delta^2} + \sum_{j=0}^{n-T} \theta^{2j} + \lim_{\delta \rightarrow \infty} \frac{2 \sum_{j=0}^{n-T} \theta^j a_{T+j}}{\delta}} \\ &= \frac{\sum_{j=0}^{n-T} \theta^{2j+k}}{\sum_{j=0}^{n-T} \theta^{2j}} \\ &= \frac{\theta^k + \theta^{2+k} + \dots + \theta^{2(n-T)+k}}{1 + \theta^2 + \dots + \theta^{2(n-T)}} \\ &= \frac{\theta^k (1 + \theta^2 + \dots + \theta^{2(n-T)})}{1 + \theta^2 + \dots + \theta^{2(n-T)}} \\ \lim_{\delta \rightarrow \infty} \hat{\rho}_k &= \theta^k \end{aligned}$$

Therefore,

$$\lim_{\delta \rightarrow \infty} \hat{\rho}_k = \theta^k, \quad k \geq 2 \quad (2.16)$$

The equations (2.13) and (2.16) prove the theorem. \square

2.3 SACF of the Errors of ARMA(1,1) model

Let the time series $\{X_t; 1, 2, \dots, n\}$ follow an ARMA(1,1) model, which is defined as

Definition 2.3 (ARMA(1, 1) Process). : $\{X_t\}$ is an ARMA(1, 1) process if $\{X_t\}$ is stationary and if for every t ,

$$X_t - \phi X_{t-1} = a_t - \theta a_{t-1}, \quad (2.17)$$

where $\{a_t\} \sim WN(0, \sigma_a^2)$ or $\{a_t\} \sim N(0, \sigma_a^2)$ and ϕ and θ are the autoregressive and moving average parameters respectively.

Assume that $\{Y_t; 1, 2, \dots, n\}$ is contaminated data, obtained by superimposing an AO of magnitude of δ at $t = T$ on $\{X_t\}$.

Theorem 2.3. Let us denote the SACF by $\hat{\rho}_k$, $k \geq 1$. Then for a fixed n

$$\lim_{\delta \rightarrow \infty} \hat{\rho}_k = \frac{-\theta^{k-1}(1-\theta^2)(\phi-\theta) + \theta^k(\phi-\theta)^2(1-\theta^{2(n-T)})}{(1-\theta^2) + (\phi-\theta)^2(1-\theta^{2(n-T)})}, \quad k \geq 1$$

Proof. By definition, when $k = 1$,

$$\hat{\rho}_1 = \frac{\sum_{t=1}^{n-1} e_t e_{t+1}}{\sum_{t=1}^n e_t^2} \quad (2.18)$$

It is known from Janhavi & Suresh [10] the errors of the contaminated series $\{e_t\}$ in the ARMA(1,1) model is

$$e_t = \begin{cases} a_t & , t < T \\ \delta + a_t & , t = T \\ -\delta\pi_j + a_t & , t = T + j, j = 1, 2, \dots, n - T \end{cases} \quad (2.19)$$

Substituting $\{e_t; 1, 2, \dots, n\}$ in terms of $\{a_t; 1, 2, \dots, n\}$ in (2.18) by using (2.19), we get

$$\hat{\rho}_1 = \frac{\sum_{t=1}^{n-1} a_t a_{t+1} - \delta \left(\sum_{j=0}^{n-T} \pi_j a_{T+j-1} + \sum_{j=0}^{n-(T-1)} \pi_j a_{T+j+1} \right) + \delta^2 \sum_{j=0}^{n-T} \pi_j \pi_{j+1}}{\sum_{t=1}^n a_t^2 + \delta^2 \sum_{j=0}^{n-T} \pi_j^2 + 2\delta \sum_{j=0}^{n-T} \pi_j a_{T+j}} \quad (2.20)$$

Now, dividing both numerator and denominator of equation (2.20) by δ^2 , we get

$$\hat{\rho}_1 = \frac{\frac{\sum_{t=1}^{n-1} a_t a_{t+1}}{\delta^2} - \frac{\delta \left(\sum_{j=0}^{n-T} \pi_j a_{T+j-1} + \sum_{j=0}^{n-(T-1)} \pi_j a_{T+j+1} \right)}{\delta^2} + \frac{\delta^2 \sum_{j=0}^{n-T} \pi_j \pi_{j+1}}{\delta^2}}{\frac{\sum_{t=1}^n a_t^2}{\delta^2} + \frac{\delta^2 \sum_{j=0}^{n-T} \pi_j^2}{\delta^2} + \frac{2\delta \sum_{j=0}^{n-T} \pi_j a_{T+j}}{\delta^2}}$$

Taking $\lim_{\delta \rightarrow \infty}$ of the above equation, for a fixed n , we get

$$\begin{aligned} \lim_{\delta \rightarrow \infty} \hat{\rho}_1 &= \lim_{\delta \rightarrow \infty} \frac{\frac{\sum_{t=1}^{n-1} a_t a_{t+1}}{\delta^2} - \frac{\delta \left(\sum_{j=0}^{n-T} \pi_j a_{T+j-1} + \sum_{j=0}^{n-(T-1)} \pi_j a_{T+j+1} \right)}{\delta^2} + \sum_{j=0}^{n-T} \pi_j \pi_{j+1}}{\frac{\sum_{t=1}^n a_t^2}{\delta^2} + \frac{\delta^2 \sum_{j=0}^{n-T} \pi_j^2}{\delta^2} + \frac{2\delta \sum_{j=0}^{n-T} \pi_j a_{T+j}}{\delta^2}} \\ &= \frac{\lim_{\delta \rightarrow \infty} \frac{\sum_{t=1}^{n-1} a_t a_{t+1}}{\delta^2} - \lim_{\delta \rightarrow \infty} \frac{\left(\sum_{j=0}^{n-T} \pi_j a_{T+j-1} + \sum_{j=0}^{n-(T-1)} \pi_j a_{T+j+1} \right)}{\delta} + \sum_{j=0}^{n-T} \pi_j \pi_{j+1}}{\lim_{\delta \rightarrow \infty} \frac{\sum_{t=1}^n a_t^2}{\delta^2} + \sum_{j=0}^{n-T} \pi_j^2 + \lim_{\delta \rightarrow \infty} \frac{2 \sum_{j=0}^{n-T} \pi_j a_{T+j}}{\delta}} \\ &= \frac{\sum_{j=0}^{n-T} \pi_j \pi_{j+1}}{\sum_{j=0}^{n-T} \pi_j^2} \\ \lim_{\delta \rightarrow \infty} \hat{\rho}_1 &= \frac{\pi_0 \pi_1 + \sum_{j=1}^{n-T} \pi_j \pi_{j+1}}{\pi_0^2 + \sum_{j=1}^{n-T} \pi_j^2} \end{aligned} \quad (2.21)$$

Note that

$$\pi_j = \begin{cases} -1 & , j = 0 \\ \theta^{j-1}(\phi - \theta) & , j > 0. \end{cases} \quad (2.22)$$

Using (2.22) in (2.21), we get

$$\begin{aligned} \lim_{\delta \rightarrow \infty} \hat{\rho}_1 &= \frac{-1(\phi - \theta) + (\phi - \theta)^2 \sum_{j=1}^{n-T} \theta^{j-1} \theta^{j+1-1}}{1 + (\phi - \theta)^2 \sum_{j=1}^{n-T} \theta^{2(j-1)}} \\ &= \frac{-(\phi - \theta) + (\phi - \theta)^2 \sum_{j=1}^{n-T} \theta^{2j-1}}{1 + (\phi - \theta)^2 \sum_{j=1}^{n-T} \theta^{2(j-1)}} \end{aligned} \quad (2.23)$$

Consider

$$\begin{aligned} \sum_{j=1}^{n-T} \theta^{2j-1} &= \theta + \theta^3 + \theta^5 + \dots + \theta^{2(n-T)-1} \\ &= \theta(1 + \theta^2 + \theta^4 + \dots + \theta^{2(n-T)}) \end{aligned}$$

Note that, the above series is a geometric series with the common ratio θ^2 . Further, $-1 < \theta < 1 \Rightarrow 0 < \theta^2 < 1$. Therefore

$$\sum_{j=1}^{n-T} \theta^{2j-1} = \frac{\theta(1 - \theta^{2(n-T)})}{(1 - \theta^2)} \quad (2.24)$$

Similarly,

$$\sum_{j=1}^{n-T} \theta^{2(j-1)} = \frac{(1 - \theta^{2(n-T)})}{(1 - \theta^2)} \quad (2.25)$$

Using (2.24) and (2.25) in (2.23)

$$\begin{aligned} \lim_{\delta \rightarrow \infty} \hat{\rho}_1 &= \frac{-(\phi - \theta) + (\phi - \theta)^2 \frac{\theta(1 - \theta^{2(n-T)})}{(1 - \theta^2)}}{1 + (\phi - \theta)^2 \frac{(1 - \theta^{2(n-T)})}{(1 - \theta^2)}} \\ \lim_{\delta \rightarrow \infty} \hat{\rho}_1 &= \frac{-(1 - \theta^2)(\phi - \theta) + \theta(\phi - \theta)^2(1 - \theta^{2(n-T)})}{(1 - \theta^2) + (\phi - \theta)^2(1 - \theta^{2(n-T)})} \end{aligned} \quad (2.26)$$

Similarly by definition, when $k \geq 2$,

$$\hat{\rho}_k = \frac{\sum_{t=1}^{n-k} e_t e_{t+k}}{\sum_{t=1}^n e_t^2} \quad (2.27)$$

Substituting $\{e_t; 1, 2, \dots, n\}$ in terms of $\{a_t; 1, 2, \dots, n\}$, it is easy to check that for $k \geq 2$,

$$\hat{\rho}_k = \frac{\sum_{t=1}^{n-k} a_t a_{t+k} - \delta \left(\sum_{j=0}^{n-T} \pi_j a_{T+j-k} + \sum_{j=0}^{n-(T-k)} \pi_j a_{T+j+k} \right) + \delta^2 \sum_{j=0}^{n-T} \pi_j \pi_{j+k}}{\sum_{t=1}^n a_t^2 + \delta^2 \sum_{j=0}^{n-T} \pi_j^2 + 2\delta \sum_{j=0}^{n-T} \pi_j a_{T+j}}$$

As before, dividing both numerator and denominator of the above equation by δ^2 , and taking $\lim_{\delta \rightarrow \infty}$, we get

$$\begin{aligned} \lim_{\delta \rightarrow \infty} \hat{\rho}_1 &= \lim_{\delta \rightarrow \infty} \frac{\frac{\sum_{t=1}^{n-k} a_t a_{t+k}}{\delta^2} - \frac{\delta \left(\sum_{j=0}^{n-T} \pi_j a_{T+j-k} + \sum_{j=0}^{n-(T-k)} \pi_j a_{T+j+k} \right)}{\delta^2} + \sum_{j=0}^{n-T} \pi_j \pi_{j+k}}{\frac{\sum_{t=1}^n a_t^2}{\delta^2} + \frac{\delta^2 \sum_{j=0}^{n-T} \pi_j^2}{\delta^2} + \frac{2\delta \sum_{j=0}^{n-T} \pi_j a_{T+j}}{\delta^2}} \\ &= \frac{\lim_{\delta \rightarrow \infty} \frac{\sum_{t=1}^{n-k} a_t a_{t+k}}{\delta^2} - \lim_{\delta \rightarrow \infty} \frac{\left(\sum_{j=0}^{n-T} \pi_j a_{T+j-k} + \sum_{j=0}^{n-(T-k)} \pi_j a_{T+j+k} \right)}{\delta} + \sum_{j=0}^{n-T} \pi_j \pi_{j+k}}{\lim_{\delta \rightarrow \infty} \frac{\sum_{t=1}^n a_t^2}{\delta^2} + \sum_{j=0}^{n-T} \pi_j^2 + \lim_{\delta \rightarrow \infty} \frac{2 \sum_{j=0}^{n-T} \pi_j a_{T+j}}{\delta}} \\ &= \frac{\sum_{j=0}^{n-T} \pi_j \pi_{j+k}}{\sum_{j=0}^{n-T} \pi_j^2} \\ \lim_{\delta \rightarrow \infty} \hat{\rho}_1 &= \frac{\pi_0 \pi_1 + \sum_{j=1}^{n-T} \pi_j \pi_{j+k}}{\pi_0^2 + \sum_{j=1}^{n-T} \pi_j^2} \end{aligned} \quad (2.28)$$

Using (2.22), in (2.28) and simplyfying, we get

$$\lim_{\delta \rightarrow \infty} \hat{\rho}_k = \frac{-\theta^{k-1}(1-\theta^2)(\phi-\theta) + \theta^k(\phi-\theta)^2(1-\theta^{2(n-T)})}{(1-\theta^2) + (\phi-\theta)^2(1-\theta^{2(n-T)})}, \quad k \geq 2 \quad (2.29)$$

The equations (2.26) and (2.29) prove the theorem. \square

3 Discussion and Conclusions

The errors $\{e_t\}$ are usually either white noise or normal process. They are independent and hence, they are uncorrelated. The main aim of this article is to shed light on the violation of this aspect. It is shown that the errors are correlated in the presence of a large AO. One large AO can cause the uncorrelated errors to be correlated, this is a serious problem. Furthermore, the pattern of the correlation between the errors depends on the time series model as well. The result of this paper is purely of theoretical interest. The consequences and applications of this work are yet to be explored. The same will be addressed in future.

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Competing Interests

Author has declared that no competing interests exist.

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