A Construction Technique for Group Divisible (*v*-1,*k*,0,1) Partially Balanced Incomplete Block Designs (PBIBDs)

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Authors' contributions

Short Research Article

Abstract

Group Divisible PBIBDs are important combinatorial structures with diverse applications. In this paper, we provided a construction technique for Group Divisible (*v*-1,*k*,0,1) PBIBDs. This was achieved by using techniques described in literature to construct Nim addition tables of order 2^n, $2 \leq n \leq 5$ and $(k^2,b,r,k,1)$Resolvable BIBDs respectively. A "block cutting" procedure was thereafter used to generate corresponding Group Divisible (*v*-1,*k*,0,1) PBIBDs from the $(k^2,b,r,k,1)$Resolvable BIBDs. These procedures were streamlined and implemented in MATLAB. The generated designs are regular with parameters(15,15,4,4,5,3,0,1);(63,63,8,8,9,7,0,1);(255,255,16,16,17,15,0,1) and (1023,1023,32,32,33,31,0,1). The MATLAB codes written are useful for generating the blocks of the designs which can be easily adapted and utilized in other relevant studies. Also, we have been able to establish a link between the game of Nim and Group Divisible (*v*-1,*k*,0,1) PBIBDs.

Keywords: Nim game; closed n-nim-regular matrices; resolvable balanced incomplete block designs; block cutting procedure; group divisible designs.

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1 Introduction

A Group Divisible PBIBD is an arrangement of $(v = mn; m, n \geq 2)$ treatments into b blocks such that each block contains $k(< v)$ distinct treatments, each treatment occurs r times and any pair of distinct treatments which are first associates occurs together in λ_1 blocks and in λ_2 blocks if they are second associates [1].

A Group divisible design is PBIBD(2) in which the underlying association scheme is group divisible [2].

A group divisible design is:

(i) Singular if $r = \lambda_1$
(ii) Semi-regular if $r > \lambda_1, rk - v\lambda_2 = 0$
(iii) Regular if $r > \lambda_1, \ rk > v\lambda_2$

Group Divisible PBIBDs are important combinatorial structures that have been extensively studied in literature. They have been found useful in the construction of new designs, plant breeding and group testing schemes [3-6]. [1] highlighted some authors who have worked on different construction techniques for this design. In addition to these works, [7] constructed a Group Divisible Design GDD($6s + 2,3,4; 2,1$) for all positive integers s, using mutually orthogonal Latin squares (MOLS) and balanced incomplete block designs (BIBDs). The generalization of this result can be found in [8].

The aim of this paper is to present a procedure for constructing and generating the blocks of Group Divisible $(v - 1, \ k, \ 0, 1)$ PBIBDs thereby establishing a link between the impartial combinatorial game, Nim and Group Divisible PBIBDs. Some definitions are provided below to enhance the flow of the paper.

Definition 1. (Balanced Incomplete Block Design (BIBD)).

A BIBD is a pair ($V, \ B$) where V is a v-set and B is a collection of b k-subsets of V (blocks) such that each element of V is contained in exactly r blocks and any 2-subset of V is contained in exactly λ blocks. The numbers $v, \ b, \ r, \ k$, and λ are the parameters of the BIBD This definition implies that v is the number of varieties, b is the number of blocks, r is the replication number of each variety, k is the size of each block and λ is the number of times each pair of varieties appear in the same block [9,10].

Definition 2. (Resolvable Balanced Incomplete Block Designs (RBIBDs))

A BIBD is said to be resolvable if its blocks can be partitioned into parallel classes [3,11,12].

Definition 3. (Partially Balanced Incomplete Block Designs (PBIBDs)).

Let X be a v-set with a symmetric association scheme defined on it. A PBIBD with m associate classes PBIBD(m) is a design based on the v-set X with b blocks, each of size k, with each treatment appearing in r blocks. Any two treatments that are ith associates appear together in λ_i blocks of the PBIBD (m). The parameters of the PBIBD(m) are $v, b, r, k, \lambda_i \ 1 \leq i \leq m)$ [2]

1.1 The game of nim

Nim is an impartial combinatorial game that has been extensively studied. The description of the game, its variants, mathematical theory including the construction of its addition table are documented in literature including [10,13-18] among others. In this paper, we utilized Nim addition tables of order $2^n, 2 \leq n \leq 5$. These tables are *closed n-nim-regular matrices*. This implies that they are Latin squares of side 2^n, and their respective mutually orthogonal Latin squares (MOLS) can be obtained.

The theory underlying the game of Nim is very important and it has been extended to all impartial combinatorial games (under normal play) and several games have been analyzed using the theory. Nim has also been linked to lexicographic codes [14,19].

In the remaining part of the paper, we have the following structure: The methodology is presented in Section 2. Results and discussion are in Section 3 while Section 4 is for the Conclusion.

2 Methodology

2.1 Theorem 1 [2,20]

Given a $(v^*, b^*, r^*, k^*, 1)$ BIBD, there exists a group divisible PBIBD $(v^* - 1, b^* - r^*, r^* - 1, k^*, r^*, k^* - 1, 0, 1)$ where $v=v*-1, b=b*-r*, r=r*-1, k=k*, m=k*-1, \lambda1=0, \lambda2=1$.

We utilized the MATLAB code in [10] to construct Nim addition tables of order $2^n, 2 \le n \le 5$ and $(k^2, b, r, k, 1)$RBIBDs respectively. A "block cutting" procedure described by [3,20] was thereafter used to implement Theorem 1 in MATLAB.

2.1.1 MATLAB code for constructing group divisible PBIBD from Nim addition Table of order$2^n, 2 \le n \le 5$

```
clear all
clc
n=input('enter n: ');


first=2^n;
second=first;
```

construct Nim addition table

```
for x=0:first
   for y=0:second
      fbin=dec2bin(x);% convert first no to base 2
      sbin=dec2bin(y);% convert second no to base 2
      lf=length(fbin);% find no of binary digits in first no
      ls=length(sbin);% find no of binary digits in second no
      if lf==ls
         n=lf;
      elseif lf~=ls % if no of bits used to represent both binary nos is no equal
         n=max(lf,ls); % determine which has the greater no of bits
      end
      fbin=dec2bin(x,n); %and represent first no
      sbin=dec2bin(y,n);%and second no using the same greater no of bits
      nimsum=zeros(1,n);
      if x~=0&&y~=0 % if neither of first and second nos is zero
         for i=1:n
            if fbin(i)==sbin(i)
               nimsum(i)=0;
            else
               nimsum(i)=1;
```

```
        end % nimsum is base 2 addition without carry
    end
    conv2str=mat2str(nimsum);% converts the binary addition to string format
    lc=length(conv2str);% find the no of elements in the string
    if (x==1&&y==1)||(x==0&&y==0)
        nimsum10=0;
    else
        nimsum10=bin2dec(conv2str(2:lc-1));%convert the nim addition to base 10
    end
    else
        nimsum10=x+y; %just add the other no to 0
    end
    nimsum101(x+1,y+1)=nimsum10;
  end
end
nimtable=nimsum101(1:first,1:second)
```

generate Mutually Orthogonal Latin Square

```
d=nimtable;

[rw,cn]=size(d);
e=zeros(((rw-1)*cn),rw);k=1;
for i=1:rw-1
e(k:i*cn,:) = [d(1,:);circshift(d(2:cn,:),i)];
k=(i*cn)+1;
end
mols=e;
```

construct Group Divisible Designs

```
n=input('confirmthe value of n: ')
bb=reshape(1:((2^n)*(2^n)),2^n,2^n)';

[rw1 cn1]=size(bb);
cc=bb';

rbibd=zeros((rw1*(rw1+1)),cn1);
count=1;
rbibd1=bb;
gdd1=rbibd1(2:end,:)
count=2;
rbibd2=cc;
gdd2=rbibd2(2:end,:)
rbibd=rbibd2;
% bibd3=zeros(rw,cn);
j=0;
count=3;
for count=3:rw1+1
    for i=1:rw1
        rbibdNEW(:,i)=circshift(rbibd(:,i),j);
        j=j-1;
    end
    rbibd=rbibdNEW;
    count=count+1;
    gdd=rbibd(2:end,:)
end
```

3 Results and Discussion

The Group divisible PBIBDs obtained for orders 2^2 and 2^3 are presented in Sections 3.1 and 3.2. The blocks are numbered using Roman numerals. Results for 2^4 and 2^5 are easily generated using the code presented in Section 2.

3.1 Group divisible PBIBD from nim addition table of order 2^2

I	5	6	7	8	IV	2	6	10	14	VII	2	7	12	13	X	2	8	10	16	XIII	2	5	12	15
II	9	10	11	12	V	3	7	11	15	VIII	3	8	9	14	XI	3	5	11	13	IV	3	6	9	16
III	13	14	15	16	VI	4	8	12	16	IX	4	5	10	15	XII	4	6	12	14	V	4	7	10	13

The parameters of this design are $v = 15, b = 15, r = 4, k = 4, m = 5, n = 3, \lambda_1 = 0, \lambda_2 = 1$.

3.2 Group divisible PBIBD from nim addition table of order 2^3

I	9	10	11	12	13	14	15	16	VIII	2	10	18	26	34	42	50	58
II	17	18	19	20	21	22	23	24	IX	3	11	19	27	35	43	51	59
III	25	26	27	28	29	30	31	32	X	4	12	20	28	36	44	52	60
IV	33	34	35	36	37	38	39	40	XI	5	13	21	29	37	45	53	61
V	41	42	43	44	45	46	47	48	XII	6	14	22	30	38	46	54	62
VI	49	50	51	52	53	54	55	56	XIII	7	15	23	31	39	47	55	63
VII	57	58	59	60	61	62	63	64	XIV	8	16	24	32	40	48	56	64

XV	2	11	20	29	38	47	56	57	XXII	2	12	22	32	34	44	54	64
XVI	3	12	21	30	39	48	49	58	XXIII	3	13	23	25	35	45	55	57
XVII	4	13	22	31	40	41	50	59	XXIV	4	14	24	26	36	46	56	58
XVIII	5	14	23	32	33	42	51	60	XXV	5	15	17	27	37	47	49	59
XIX	6	15	24	25	34	43	52	61	XXVI	6	16	18	28	38	48	50	60
XX	7	16	17	26	35	44	53	62	XXVII	7	9	19	29	39	41	51	61
XXI	8	9	18	27	36	45	54	63	XXVIII	8	10	20	30	40	42	52	62

XXIX	2	13	24	27	38	41	52	63	XXXVI	2	14	18	30	34	46	50	62
XXX	3	14	17	28	39	42	53	64	XXXVII	3	15	19	31	35	47	51	63
XXXI	4	15	18	29	40	43	54	57	XXXVIII	4	16	20	32	36	48	52	64
XXXII	5	16	19	30	33	44	55	58	XXXIX	5	9	21	25	37	41	53	57
XXXIII	6	9	20	31	34	45	56	59	XL	6	10	22	26	38	42	54	58
XXXIV	7	10	21	32	35	46	49	60	XLI	7	11	23	27	39	43	55	59
XXXV	8	11	22	25	36	47	50	61	XLII	8	12	24	28	40	44	56	60

XLIII	2	15	20	25	38	43	56	61	L	2	16	22	28	34	48	54	60
XLIV	3	16	21	26	39	44	49	62	LI	3	9	23	29	35	41	55	61
XLV	4	9	22	27	40	45	50	63	LII	4	10	24	30	36	42	56	62
XLVI	5	10	23	28	33	46	51	64	LIII	5	11	17	31	37	43	49	63
XLVII	6	11	24	29	34	47	52	57	LIV	6	12	18	32	38	44	50	64
XLVIII	7	12	17	30	35	48	53	58	LV	7	13	19	25	39	45	51	57
XLIX	8	13	18	31	36	41	54	59	LVI	8	14	20	26	40	46	52	58

LVII	2	9	24	31	38	45	52	59
LVIII	3	10	17	32	39	46	53	60
LIX	4	11	18	25	40	47	54	61
LX	5	12	19	26	33	48	55	62
LXI	6	13	20	27	34	41	56	63
LXII	7	14	21	28	35	42	49	64
LXIII	8	15	22	29	36	43	50	57

The parameters of this design are $v = 63, b = 63, r = 8, k = 8, m = 9, n = 7, \lambda_1 = 0, \lambda_2 = 1$.

The Group Divisible PBIBDs constructed via this approach are all regular, that is $r > \lambda_1$ and $rk > v\lambda_2$. A summary of the results is presented in Table 1.

Table 1. Parameters of group divisible $(v - 1, \ k, 0, 1)$PBIBD obtained via nim addition table

Nim addition Table of order 2^n	Parameters of the $(k^2, b, r, k, 1)$RBIBD	Parameters of the group divisible $(v - 1, \ k, 0, 1)$PBIBD
2^2	16, 20, 5, 4,1	15, 15, 4, 4, 5, 3, 0, 1
2^3	64, 72, 9, 8, 1	63, 63, 8, 8, 9, 7, 0, 1
2^4	256, 272, 17, 16, 1	255, 255, 16, 16, 17, 15, 0, 1
2^5	1024, 1056, 33, 32, 1	1023, 1023, 32, 32, 33, 31, 0, 1

4 Conclusion

Group Divisible PBIBDs are important combinatorial structures with applications in diverse fields. In this paper, Nim addition tables of order 2^n, $2 \leq n \leq 5$ and $(k^2, b, r, k, 1)$RBIBDs respectively were constructed using techniques described in [10]. From the $(k^2, b, r, k, 1)$RBIBDs, a "block cutting" procedure was used to generate corresponding Group Divisible $(v-1, \ k, \ 0, 1)$ PBIBDs. All these procedures were implemented in MATLAB. The parameters of the Group Divisible $(v-1, \ k, \ 0, 1)$ PBIBDs generated are $(15, 15, 4, 4, 5, 3, 0; (63, 63, 8, 8, 9, 7, 0, 1); (255, 255, 16, 16, 17, 15, 0, 1);$ and $(1023, 1023, 32, 32, 33, 31, 0, 1)$ and they are all regular. The MATLAB codes written are useful for generating the blocks of the designs which can be easily adapted and utilized in other relevant studies. Also, we have been able to establish a link between the game of Nim and Group Divisible $(v-1, \ k, \ 0, 1)$ PBIBDs.

Competing Interests

Authors have declared that no competing interests exist.

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