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# Solving Nonlinear Systems Using Fourth Order Iterative Method

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***Research Article***

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## Abstract

In this paper, an efficient decomposition method is constructed and used for solving system of nonlinear equations. These methods based on the modified homotopy technique of Noor [1]. This technique is revised to solve the system of nonlinear equations. Our approach yields third and fourth order iterative methods which are more efficient than their classical counterparts such as Newton's, Chebychev's and Halley's methods.

Keywords: Homotopy method, perturbation method, System of nonlinear equations, Iterative methods, Newton's method.

## 1 Introduction

Homotopy perturbation methods (HPM) play a very important role for solving several mathematical problems, for instance, linear and nonlinear system of equations, differential equations and integral equations [1-10]. The basic idea of HPM is to simplify the difficult equation systems by converting them into either linear or nonlinear system of equations so that they can be easily solved. In the recent years, HPM attracts the attention of researchers, because solutions by this method offer a high degree of accuracy and convergency [11-19].

He [20] suggested an iterative method for solving the nonlinear equations by rewriting the given nonlinear equation as a coupled system of equations. This technique has been used by Chun [21] and Noor et al. [22,23] to suggest some higher order convergent iterative methods for solving nonlinear equations. Golbabai, and Javidi [24] applied HPM for solving system of nonlinear equations in two dimensions by expanding the variables into Taylor series. Noor et al. [1] modified homotopy perturbation method by combining the homotopy analysis method and HPM, for solving nonlinear equations in one dimension. In this research we revised this technique to solve system of nonlinear equations of *n*-dimension with *n*-variables. Some illustrative examples have been presented, to demonstrate the accuracy of proposed methods and the results are compared with those derived from the previous methods.

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# 2 Iterative Methods

If we have a system of nonlinear equations in the form:

$$\begin{gathered} f_1(x_1, x_2, \cdots, x_n) = 0, \\ f_2(x_1, x_2, \cdots, x_n) = 0, \\ \vdots \\ f_n(x_1, x_2, \cdots, x_n) = 0, \end{gathered}$$

where $f_i : R^n \to R$ are differentiable up to any desired order [25], this system can alternatively be represented by defining a functional $F : R^n \to R^n$, as follows:

$$F(x_1, x_2, \ldots x_n) = [f_1(x_1, x_2, \ldots x_n), \ldots, f_n(x_1, x_2, \ldots x_n)]^T .$$

The previous system can be written in the vector notation as follows:

$$F(\mathrm{x}) = 0 \tag{1}$$

If we assume that $\mathrm{x}^*$ is a simple root of Eq.(1) and $\mathrm{x}_0$ is an initial guess sufficiently close to $\mathrm{x}^*$. We can apply the technique of He [20] and rewrite the nonlinear Eq.(1) as a coupled system using the Taylor's series as follows:

$$f_k(\mathrm{x}_0) + \frac{1}{1!}\Big[\sum_{i=1}^{n} f_{k,i}(\mathrm{x}_0)(x_i - x_i^{(0)})\Big] + \frac{1}{2!}\Big[\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_i^{(0)})^T f_{k,ij}(\mathrm{x}_0)(x_j - x_j^{(0)})\Big] + g_k(\mathrm{x}) = 0 \tag{2}$$

$$g_k(\mathrm{x}) = f_k(\mathrm{x}) - f_k(\mathrm{x}_0) - \frac{1}{1!}\Big[\sum_{i=1}^{n} f_{k,i}(\mathrm{x}_0)(x_i - x_i^{(0)})\Big] - \frac{1}{2!}\Big[\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_i^{(0)})^T f_{k,ij}(\mathrm{x}_0)(x_j - x_j^{(0)})\Big] \tag{3}$$

where $k = 1, 2, \ldots, n$, $f_{k,i} = \dfrac{\partial f_k}{\partial x_i}$, $f_{k,ij} = \dfrac{\partial^2 f_k}{\partial x_i \partial x_j}$ and $\mathrm{x}_0 = [x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}]^T$ is the initial approximation for the zero of Eq.(1).

Matrices of first and second partial derivatives of the function *f* appearing in equation (2) are the Jacobian $J$ and Hessian $H$ matrix of this function respectively. Putting Eq.(2) in matrix notation

$$F(\mathrm{x}_0) + J(\mathrm{x}_0)[\mathrm{x} - \mathrm{x}_0] + \frac{1}{2!}\sum_{i=1}^{n} e_i \otimes [\mathrm{x} - \mathrm{x}_0]^T H_i(\mathrm{x}_0)[\mathrm{x} - \mathrm{x}_0] + G(\mathrm{x}) = 0 \tag{4}$$

$$G(\mathrm{x}) = F(\mathrm{x}) - F(\mathrm{x}_0) - J(\mathrm{x}_0)[\mathrm{x} - \mathrm{x}_0] - \frac{1}{2!}\sum_{i=1}^{n} e_i \otimes [\mathrm{x} - \mathrm{x}_0]^T H_i(\mathrm{x}_0)[\mathrm{x} - \mathrm{x}_0] \tag{5}$$

where $H_i$ is the Hessian matrix of the function $f_i$ , $\otimes$ is the Kronecker product and $e_i$ is a $n \times 1$ vector of zeroes except for a 1 in the position $i$, We can rewrite Eq.(5) in the following form

$$\mathrm{x} = \mathrm{x}_0 - [J(\mathrm{x}_0)]^{-1}\left\{G(\mathrm{x}) + F(\mathrm{x}_0) + \frac{1}{2!}\sum_{i=1}^{n} e_i \otimes [\mathrm{x}-\mathrm{x}_0]^T H_i(\mathrm{x}_0)[\mathrm{x}-\mathrm{x}_0]\right\} \tag{6}$$

From Eq.(5)

$$G(\mathrm{x}^{(0)}) = F(\mathrm{x}^{(0)}) \tag{7}$$

One can rewrite Eq.(6) as:

$$\mathrm{x} = C + \hbar N(\mathrm{x}) \tag{8}$$

where

$$C = \mathrm{x}_0 - [J(\mathrm{x}_0)]^{-1} F(\mathrm{x}_0) \tag{9}$$

and

$$N(\mathrm{x}) = -[J(\mathrm{x}_0)]^{-1}\{G(\mathrm{x}) + \frac{1}{2!}\sum_{i=1}^{n} e_i \otimes [\mathrm{x}-\mathrm{x}_0]^T H_i(\mathrm{x}_0)[\mathrm{x}-\mathrm{x}_0]\} \tag{10}$$

Now, we construct a homotopy $H(\bar{\mathrm{x}}, p, T): R^n \times [0,1] \to R$ which satisfies

$$H(\bar{\mathrm{x}}, p, T) = (1-p)[L(\bar{\mathrm{x}}) - L(\mathrm{x}_0)] + p[L(\bar{\mathrm{x}}) - C - \hbar N(\bar{\mathrm{x}})] - \hbar p^2 (1-p)\Gamma = 0 \tag{11}$$

or

$$H(\bar{\mathrm{x}}, p, T) = L(\bar{\mathrm{x}}) - L(\mathrm{x}_0) + pL(\mathrm{x}_0) - p[C + \hbar N(\bar{\mathrm{x}})] - \hbar p^2 (1-p)\Gamma = 0 \tag{12}$$

where $p \in [0,1]$ is an embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, and $\Gamma$ is an arbitrary auxiliary operator. We emphasize that we have great freedom to select the initial guess, the auxiliary parameter $\hbar$ and the auxiliary operator $\Gamma$.

from Eqs. (11) and (12), we have

$$H(\bar{\mathrm{x}}, 0, T) = L(\bar{\mathrm{x}}) - L(\mathrm{x}_0) = \bar{\mathrm{x}} - C = 0 \tag{13}$$

or

$$H(\bar{\mathrm{x}}, 1, T) = L(\bar{\mathrm{x}}) - C - \hbar N(\bar{\mathrm{x}}) = \bar{\mathrm{x}} - C - \hbar N(\bar{\mathrm{x}}) = 0 \tag{14}$$

The parameter $p$ monotonically increases from zero to unity as the trivial problem (13), is continuously deformed to the original problem (14). The changing process of $p$ from zero to unity is called deformation. Now, suppose that the solution of Eqs. (11) and (12) can be expressed as a power series in $p$:

$$\bar{x} = x^{(0)} + \sum_{k=1}^{\infty} p^k x^{(k)} \tag{15}$$

And hence, the approximate solution of Eq.(8), can be in the form:

$$\bar{x} = \lim_{p \to 1} \bar{x} = x^{(0)} + \sum_{k=1}^{\infty} x^{(k)} \tag{16}$$

For the application of HPM to Eq.(1), we can write Eq.(12) by expanding $N(x)$ into a Taylor series around $x^{(0)}$ as follows:

$$-p\hbar\left\{N(x^{(0)}) + N^J(x^{(0)})[\bar{x} - x^{(0)}] + \frac{1}{2!}\sum_{i=1}^{n} e_i \otimes [\bar{x} - x^{(0)}]^T N_i^H(x^{(0)})[\bar{x} - x^{(0)}] + \cdots\right\}$$
$$+\bar{x} - C - \hbar p^2(1-p)\Gamma = 0 \tag{17}$$

Substituting Eq.(15) into Eq.(17), and equating the coefficients of the identical power of*p*, we get

$$p^0 : x^{(0)} = C \tag{18}$$

$$p^1 : x^{(1)} = \hbar N(x^{(0)}) \tag{19}$$

$$p^2 : x^{(2)} = \hbar N^J(x^{(0)})[x^{(1)}] + \hbar\Gamma \tag{20}$$

$$p^3 : x^{(3)} = \hbar N^J(x^{(0)})[x^{(2)}] + \frac{\hbar}{2!}\sum_{i=1}^{n} e_i \otimes [x^{(1)}]^T N_i^H(x^{(0)})[x^{(1)}] - \hbar\Gamma \tag{21}$$

Taking $x^{(2)} = 0$, we get

$$\Gamma = -N^J(x^{(0)})[x^{(1)}]$$

$$x^{(3)} = \frac{\hbar}{2!}\sum_{i=1}^{n} e_i \otimes [x^{(1)}]^T N_i^H(x^{(0)})[x^{(1)}] + \hbar N^J(x^{(0)})[x^{(1)}]$$

From Eqs. (5), and (10), we have the Jacobian and Hessian matrix in the form

$$N^J(x^{(0)}) = I - [J(x_0)]^{-1} J(x^{(0)})$$

$$N_i^H(x^{(0)}) = -[J(x_0)]^{-1} H_i(x^{(0)})$$

from equations (9) and (18) the first approximation becomes

$$x^{(0)} = x_0 - [J(x_0)]^{-1} F(x_0)$$

which gives the following iterative method.

**Algorithm 2.1**For a given $x_0$, compute approximate solution $x_{n+1}$ by the iterative scheme.

$$x_{n+1} = x_n - [J(x_n)]^{-1} F(x_n)$$

this is the well-known Newton method (NM).

For $n = 1$

$$x \approx x^{(0)} + x^{(1)} = C + \hbar N(x^{(0)})$$

$$= x_0 - [J(x_0)]^{-1} F(x_0) - \hbar [J(x_0)]^{-1} \{G(x^{(0)}) + \frac{1}{2!} \sum_{i=1}^{n} e_i \otimes [x^{(0)} - x_0]^T H_i(x_0) [x^{(0)} - x_0]\}$$

For $\hbar = 1$ and $G(x^{(0)}) = 0$, we get the following iterative method.

**Algorithm 2.2** For a given $x_0$, compute approximate solution $x_{n+1}$ by the iterative scheme.

$$y_n = x_n - [J(x_n)]^{-1} F(x_n)$$

$$x_{n+1} = y_n - \frac{1}{2!} [J(x_n)]^{-1} \sum_{i=1}^{n} e_i \otimes [y_n - x_n]^T H_i(x_n) [y_n - x_n]$$

For $n = 3$

$$x \approx x^{(0)} + x^{(1)} + x^{(3)} = C + \hbar N(x^{(0)}) + \hbar N^J(x^{(0)})[x^{(1)}] + \frac{\hbar}{2!} \sum_{i=1}^{n} e_i \otimes [x^{(1)}]^T N_i^H(x^{(0)})[x^{(1)}]$$

$$= x_0 - [J(x_0)]^{-1} F(x_0) - [J(x_0)]^{-1} \{G(x^{(0)}) + \frac{1}{2!} \sum_{i=1}^{n} e_i \otimes [x^{(0)} - x_0]^T H_i(x_0)[x^{(0)} - x_0]\}$$

$$+ \hbar \{I - [J(x_0)]^{-1} J(x^{(0)})\}[x^{(1)}] - \frac{\hbar}{2!} \sum_{i=1}^{n} e_i \otimes [x^{(1)}]^T \{[J(x_0)]^{-1} H_i(x^{(0)})\}[x^{(1)}]$$

For $\hbar = 1$ and $H_i(x_0) = 0$, this formulation allows us the following new iterative method.

**Algorithm 2.3** For a given $x_0$, compute approximate solution $x_{n+1}$ by the iterative scheme.

$$y_n = x_n - [J(x_n)]^{-1} F(x_n)$$

$$\mathrm{x}_{n+1} = \mathrm{y}_n - 2[J(\mathrm{x}_n)]^{-1}F(\mathrm{y}_n) + \{[J(\mathrm{x}_n)]^{-1}J(\mathrm{y}_n)\}\{[J(\mathrm{x}_n)]^{-1}F(\mathrm{y}_n)\}$$
$$-\frac{1}{2!}\sum_{i=1}^{n} e_i \otimes \{[J(\mathrm{x}_n)]^{-1}F(\mathrm{y}_n)\}^T \{[J(\mathrm{x}_n)]^{-1}H_i(\mathrm{y}_n)\}\{[J(\mathrm{x}_n)]^{-1}F(\mathrm{y}_n)\}$$

## 3 Analysis of Convergence

Before exploring the local convergence properties of the Algorithm 2.2, we will state the following result on Taylor's expansion of vector functions [26].

**Lemma 1** *Assume that* $F: D \subset R^n \to R^n$ *be a* $C^p$ *function defined on*

$D = \{x : \| x - a \| < r\}$; *then for any* $v \le r$ *, the following expression holds,*

$$F(a+v) = F(a) + F'(a)v + \frac{1}{2}F''(a)vv + \cdots + \frac{1}{p!}F^{(p)}(a)v\cdots v + R_p,$$

*where*

$$\| R_p \| \le \sup_{x\in D} \frac{\|v\|^p}{p!} \| F^{(p)}(x) - F^{(p)}(a) \|.$$

We can now state and prove the main result.

**Theorem 1** *Let* $F: D \subset R^n \to R^n$ *be a* $C^4$ *function in an open convex set* $D \subset R^n$. *Assume that there exists an* $\alpha \in D$ *such that* $F(\alpha) = 0$ *and* $F'(\alpha)^{-1}$ *exists. Then there exists an* $\varepsilon > 0$ *such that for any* $x_0 \in U(\alpha, \varepsilon)$ *the sequence generated by Algorithm 2.2 is well defined and converges to the zero α of F, and the process has order three.*

*Proof* We introduce the notations

$$\delta = x_{n+1} - \alpha,\ \varepsilon = x_n - \alpha,\ H(x_n) = F'(x_n)^{-1}F(x_n),\ B(x_n) = F'(x_n)^{-1}F''(x_n).$$

From Algorithm 2.2,we obtain

$$\delta = (\varepsilon - H(x_n)) + \frac{1}{2}B(x_n)H(x_n)H(x_n) \tag{22}$$

Using $H(\alpha) = I$ , $F(\alpha) = 0$ and Lemma 1 we represent $H(x_n)$ by Taylor expansion as:

$$\begin{aligned} H(x_n) &= H(\alpha) + H'(\alpha)\varepsilon + \frac{1}{2}H''(\alpha)\varepsilon\varepsilon + \frac{1}{6}H'''(\alpha)\varepsilon\varepsilon\varepsilon + O(\|\varepsilon\|^4) \\ &= \varepsilon + \frac{1}{2}H''(\alpha)\varepsilon\varepsilon + \frac{1}{6}H'''(\alpha)\varepsilon\varepsilon\varepsilon + O(\|\varepsilon\|^4) \end{aligned} \tag{23}$$

Similarly, we express:

$$\begin{aligned} B(x_n) &= B(\alpha) + B'(\alpha)\varepsilon + \frac{1}{2}B''(\alpha)\varepsilon\varepsilon + \frac{1}{6}B'''(\alpha)\varepsilon\varepsilon\varepsilon + O(\|\varepsilon\|^4) \\ &= -H''(\alpha) + B'(\alpha)\varepsilon + \frac{1}{2}B''(\alpha)\varepsilon\varepsilon + \frac{1}{6}B'''(\alpha)\varepsilon\varepsilon\varepsilon + O(\|\varepsilon\|^4) \end{aligned} \tag{24}$$

where $B(\alpha) = -H''(\alpha)$

Using (22) , (23) and (24), we obtain

$$\begin{aligned} \delta = {} & [\varepsilon - (\varepsilon + \frac{1}{2}H''(\alpha)\varepsilon\varepsilon + \frac{1}{6}H'''(\alpha)\varepsilon\varepsilon\varepsilon)] \\ & + \frac{1}{2}\Bigg[\left(-H''(\alpha) + B'(\alpha)\varepsilon + \frac{1}{2}B''(\alpha)\varepsilon\varepsilon + \frac{1}{6}B'''(\alpha)\varepsilon\varepsilon\varepsilon\right) \\ & \left(\varepsilon + \frac{1}{2}H''(\alpha)\varepsilon\varepsilon + \frac{1}{6}H'''(\alpha)\varepsilon\varepsilon\varepsilon\right) \\ & \left(\varepsilon + \frac{1}{2}H''(\alpha)\varepsilon\varepsilon + \frac{1}{6}H'''(\alpha)\varepsilon\varepsilon\varepsilon\right)\Bigg] + O(\|\varepsilon\|^4) \end{aligned}$$

Further, we obtain

$$\delta = \left(-\frac{1}{6}H'''(\alpha) + B'(\alpha) - \frac{1}{2}H''(\alpha)H''(\alpha)\right)\varepsilon\varepsilon\varepsilon + O(\|\varepsilon\|^4)$$

The latter expression implies the assertions in the statement of Theorem 1, and this completes the proof.

## 4 Numerical Examples

In order to demonstrate the performance of the introduced iterative methods 2.2, 2.3 as a novel solver for systems of nonlinear equations, four different problems were selected as famous test problems found in literatures. We present the results of our comparison of methods obtained from Algorithm 2.2-2.3, which we call, (M1, M2) with the classical Newton's method (NM), the third-order Hafiz and Bahgat method (HBM) [27], Darvishi method (DAM) [28] and Khirallah and Hafiz method (KHM1) [29] algorithms, respectively, take the form

$$x_{n+1} = x_n - 12\left[J(x_n) + 10J(w_n) + J(y_n)\right]^{-1} F(x_n),$$
$$x_{n+1} = x_n - 2\left[F'(x_n) + F'(y_n)\right]^{-1} F(x_n),$$
$$x_{i+1} = x_i - 6[F'(x_i) + 4F'(w_n) + F'(y_i)]^{-1} F(x_i)$$

where

$$y_n = x_n - J^{-1}(x_n)F(x_n), \qquad w_n = \frac{x_i + y_i}{2}.$$

Here, numerical results are performed by Maple 15 with 200 digits. The following stopping criteria is used

$$\| x_{n+1} - x_n \| + \| F(x_n) \| < 10^{-15}$$

and the computational order of convergence (COC) can be taken in the following form:

$$\mathrm{COC} \approx \frac{\ln\left(\| x_{n+1} - x_n \| / \| x_n - x_{n-1} \|\right)}{\ln\left(\| x_n - x_{n-1} \| / \| x_{n-1} - x_{n-2} \|\right)}$$

Table 2 shows the number of iterations and the computational order of convergence (COC). $\| x_{n+1} - x_n \|$ and the norm of the function $F(x_n)$ is also shown for various methods.

**Example 1**. In the case of one dimension, consider the following nonlinear functions [29], $f_1(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5,$ with $x_0 = -3$ and $f_2(x) = e^{x^2+7x-30} - 1$ with $x_0 = 4.$

**Table 1. Numerical results for Example 1**

| Methods | IT | COC | $\| x_{n+1} - x_n \|$ | $\| F(x_n) \|$ |
|---|---|---|---|---|
| $F_1$, $x_0$ =-3. | | | | |
| NM | 15 | 2.0000 | 0.4625E-27 | 0.6524E-53 |
| HBM | 10 | 3.0000 | 0.1057E-37 | 0.4814E-112 |
| DAM | 10 | 2.9992 | 0.3706E-17 | 0.3367E-50 |
| KHM1 | 10 | 3.0000 | 0.4530E-32 | 0.4261E-95 |
| M1 | 10 | 2.9999 | 0.2535E-23 | 0.8303E-69 |
| M2 | 9 | 3.9984 | 0.8755E-24 | 0.1213E-93 |
| $F_2$, x0 =4 | | | | |
| NM | 20 | 2.0000 | 0.7040E-20 | 0.4237E-38 |
| HBM | 13 | 2.9998 | 0.1593E-24 | 0.2081E-71 |
| DAM | 14 | 2.9998 | 0.1864E-24 | 0.4872E-71 |
| KHM1 | 13 | 2.9993 | 0.9881E-20 | 0.5425E-57 |
| M1 | 14 | 3.0000 | 0.2311E-32 | 0.9204E-95 |
| M2 | 12 | 3.9785 | 0.1771E-16 | 0.1091E-62 |

**Example 2.** In the case of two dimension, consider the following systems of nonlinear equations [30],

$$F_3(\mathrm{x}) = \begin{cases} f_1(x,y) = x^2 - 10x + y^2 + 8 = 0 \\ f_2(x,y) = xy^2 + x - 10y + 8 = 0 \end{cases}, \qquad (x_0, y_0) = (0.8, 0.8).$$

$$F_4(\mathrm{x}) = \begin{cases} f_1(x,y) = x^4 y - xy + 2x - y - 1 = 0 \\ f_2(x,y) = ye^{-x} + x - y - e^{-1} = 0 \end{cases}, \qquad (x_0, y_0) = (2, 2).$$

**Table 2. Numerical results for Example2**

| **Methods** | **IT** | **COC** | $\Vert \mathrm{x}_{n+1} - \mathrm{x}_n \Vert$ | $\Vert F(\mathrm{x}_n) \Vert$ |
|---|---|---|---|---|
| $F_3$, $X_0$=(0.8, 0.8) | | | | |
| NM | 5 | 1.999 | 0.3640E-19 | 0.2498E-34 |
| HBM | 4 | 2.999 | 0.4110E-29 | 0.4359E-88 |
| DAM | 4 | 2.996 | 0.2221E-31 | 0.4443E-95 |
| KHM1 | 4 | 2.999 | 0.1669E-29 | 0.2709E-89 |
| M1 | 4 | 2.998 | 0.4538E-26 | 0.1089E-78 |
| M2 | 4 | 3.999 | 0.6649E-60 | 0.2000E-198 |
| $F_4$, $X_0$=(2, 2) | | | | |
| NM | 9 | 2.000 | 0.7530E-18 | 0.2301E-35 |
| HBM | 6 | 3.002 | 0.1189E-17 | 0.1021E-52 |
| DAM | 7 | 3.000 | 0.1177E-42 | 0.1378E-127 |
| KHM1 | 6 | 3.001 | 0.7024E-17 | 0.2267E-50 |
| M1 | 7 | 3.000 | 0.3197E-38 | 0.3037E-114 |
| M2 | 6 | 4.001 | 0.5571E-26 | 0.5226E-103 |

**Example 3.** In the case of three dimension, consider the following systems of nonlinear equations [31].

$$F_5(\mathrm{x}) = \begin{cases} f_1(x,y,z) = 15x + y^2 - 4z - 13 = 0 \\ f_2(x,y,z) = x^2 + 10y - e^{-z} - 11 = 0 \\ f_3(x,y,z) = y^3 - 25z + 22 = 0 \end{cases}, \quad X_0 = (1.2, -1.8, 0.1)$$

$$F_6(\mathrm{x}) = \begin{cases} f_1(x,y,z) = 3x - \cos(yz) - 0.5 = 0 \\ f_2(x,y,z) = x^2 - 81(y+0.1)^2 + \sin z + 1.06 = 0 \\ f_3(x,y,z) = e^{-xy} + 20z + \dfrac{10\pi - 3}{3} = 0 \end{cases}, \quad X_0 = (0.1, 0.1, -0.4).$$

**Table 3. Numerical results for Example 3**

| Methods | IT | COC | $\Vert x_{n+1} - x_n \Vert$ | $\Vert F(x_n) \Vert$ |
|---|---|---|---|---|
| $F_5$, $X_0$=(1.2, -1.8, 0.1) | | | | |
| NM | 6 | 1.8225 | 0.2888E-19 | 0.9630E-40 |
| HBM | 4 | 2.8983 | 0.1597E-20 | 0.3115E-64 |
| DAM | 5 | 3.0860 | 0.6181E-42 | 0.1141E-126 |
| KHM1 | 4 | 3.1514 | 0.1439E-22 | 0.3072E-70 |
| M1 | 5 | 3.1073 | 0.6875E-31 | 0.3083E-93 |
| M2 | 4 | 4.1222 | 0.1114E-37 | 0.1436E-153 |
| $F_6$, $X_0$=(0.1, 0.1, -0.4) | | | | |
| NM | 6 | 2.0000 | 0.8901E-16 | 0.6413E-30 |
| HBM | 5 | 2.9999 | 0.3360E-39 | 0.1538E-115 |
| DAM | 5 | 2.9999 | 0.3409E-39 | 0.1606E-115 |
| KHM1 | 5 | 2.9999 | 0.3370E-39 | 0.1552E-115 |
| M1 | 4 | 2.9999 | 0.1277E-31 | 0.1691E-92 |
| M2 | 4 | 3.9789 | 0.4140E-22 | 0.2984E-85 |

**Example 4**. Consider the kinematic synthesis mechanism for automotive steering. This problem is originally described in [32]. The Ackerman steering mechanism is a four-bar mechanism for steering four wheel vehicles. When a vehicle turns, the steered wheels need to be angled so that they are both 90º with respect to a certain line. This means that the wheels will have to be at different angles with respect to the non-steering wheels.

The Ackerman design arranges the wheels automatically by moving the steering pivot inward. Pramanik [32] stated that “the Ackerman design reveals progressive deviations from ideal steering with increasing ranges of motion“. Pramanik instead considered a six-member mechanism. This produces the system of equations given, for i = 1, 2, 3, by

$$\begin{aligned} F_7(x) = G_i(\psi_i, \phi_i) = {} & \left[E_i\left(y\sin(\psi_i) - z\right) - F_i\left(y\sin(\phi_i) - z\right)\right]^2 + \\ & \left[F_i\left(1 + y\cos(\phi_i)\right) - E_i\left(y\cos(\psi_i) - 1\right)\right]^2 - \\ & \left[\left(1 + y\cos(\phi_i)\right)\left(y\sin(\psi_i) - z\right)x - \right. \\ & \left.\left(y\sin(\phi_i) - z\right)\left(y\cos(\phi_i) - z\right)x\right]^2, \end{aligned}$$

where

$$E_i = y\left(\cos(\phi_i) - \cos(\phi_0)\right) - yz\left(\sin(\phi_i) - \sin(\phi_0)\right) - \left(y\sin(\phi_i) - z\right)x,$$

and

$$F_i = -y\cos(\psi_i) - yz\sin(\psi_i) + y\cos(\psi_0) + xz + \left(z - x\right)y\sin(\psi_0).$$

When the angles $\psi_i$ and $\phi_i$ are given as in Table 4, there are two roots for the system in the domain $[0.5, 1]^3$.

**Table 4. Angular data (in radians) for automotive steering problem**

| *i* | $\psi_i$ | $\phi_i$ |
|---|---|---|
| 0 | 1.3954170041747090114 | 1.7461756494150842271 |
| 1 | 1.7444828545735749268 | 2.0364691127919609051 |
| 2 | 2.0656234369405315689 | 2.2390977868265978920 |
| 3 | 2.4600678478912500533 | 2.4600678409809344550 |

**Table 5. Numerical results for Example 4**

| **Methods** | **IT** | **COC** | $\|\| x_{n+1} - x_n \|\|$ | $\|\| F(x_n) \|\|$ |
|---|---|---|---|---|
| $F_7$, $X_0$=(0.6, 0.9, 0.8) | | | | |
| NM | 8 | 1.9994 | 0.1340E-23 | 0.1556E-48 |
| HBM | 5 | 2.8564 | 0.3000E-16 | 0.2876E-49 |
| DAM | 6 | 2.9489 | 0.1572E-34 | 0.3138E-104 |
| KHM1 | 5 | 2.8204 | 0.1390E-15 | 0.2843E-47 |
| M1 | 6 | 3.0175 | 0.6865E-35 | 0.5323E-105 |
| M2 | 5 | 4.221 | 0.3488E-19 | 0.8773E-77 |
| $F_7$, $X_0$=(0.6, 0.9, 0.9) | | | | |
| NM | 6 | 2.0393 | 0.8977E-28 | 0.1532E-56 |
| HBM | 4 | 2.7119 | 0.4229E-24 | 0.9155E-73 |
| DAM | 4 | 3.2442 | 0.5195E-22 | 0.1874E-65 |
| KHM1 | 4 | 2.8026 | 0.4034E-27 | 0.2572E-81 |
| M1 | 4 | 3.264 | 0.4021E-16 | 0.6222E-49 |
| M2 | 4 | 4.3400 | 0.1964E-37 | 0.9934E-148 |

## 5 Conclusions

Our study presents a family of third and fourth order iterative methods for solving systems of nonlinear equations. The numerical examples show in general that our method are very effective and efficient and provide highly accurate results in a less number of iterations as compared to some well-known methods when the initial value $x_0$ is good chosen. It is an open problem to determine the most appropriate choice of the initial guess.

## Competing Interests

Authors have declared that no competing interests exist.

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